Time-Uniform Self-Normalized Concentration for Vector-Valued Processes

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Abstract

Self-normalized processes arise naturally in many statistical tasks. While self-normalized concentration has been extensively studied for scalar-valued processes, there is less work on multidimensional processes outside of the sub-Gaussian setting. In this work, we construct a general, self-normalized inequality for $\mathbb{R}^d$-valued processes that satisfy a simple yet broad "sub-ψ" tail condition, which generalizes assumptions based on cumulant generating functions. From this general inequality, we derive an upper law of the iterated logarithm for sub-ψ vector-valued processes, which is tight up to small constants. We demonstrate applications in prototypical statistical tasks, such as parameter estimation in online linear regression and auto-regressive modeling, and bounded mean estimation via a new (multivariate) empirical Bernstein concentration inequality.

1 Introduction

Concentration inequalities are employed in many disparate mathematical fields. In particular, time-uniform martingale concentration has proven itself a critcal tool in advancing research areas such as multi-armed bandits [35, 2, 42], differential privacy [60, 59], Bayesian learning [14], and online convex optimization [43, 34]. While martingale concentration inequalities have historically been proved in a largely case-by-case manner, recently Howard et al. [31] provided a unified framework for constructing time-uniform concentration inequalities. By introducing a single "sub-ψ" assumption that carefully controls the tail behavior of martingale increments, Howard et al. [31, 32] prove a master theorem that recovers (in fact improves) many classical examples of concentration inequalities, for example those of Blackwell [10], Hoeffding [30], Freedman [28], Azuma [6], de la Peña et al. [20].

Despite the generality of the framework of Howard et al. [31, 32], their results have not been extended to understanding the growth of "self-normalized" vector-valued processes. If

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$(S_t)_{t\geq 0}$ is a process evolving in $\mathbb{R}^d$ and $(V_t)_{t\geq 0}$ is a process of $d \times d$ positive semi-definite matrices measuring the "accumulated variance" of $(S_t)_{t\geq 0}$, self-normalized concentration aims to control the growth of the normalized process $(\|V_t^{-1/2}S_t\|)_{t\geq 0}$. Self-normalized processes naturally arise in a variety of common statistical tasks, examples of which include regression problems [40, 41, 8] and contextual bandit problems [2, 12]. As such, any advances in self-normalized concentration for vector-valued processes could directly yield improvements in methodology and analysis of foundational statistical algorithms.

In this work, we provide a new, general approach for constructing self-normalized concentration inequalities. By naturally generalizing the sub-ψ condition of Howard et al. [31, 32] to d-dimensional spaces, we are able to construct a single "master" theorem that provides time-uniform, self-normalized concentration under a variety of noise settings. We prove our results by first constructing a time-uniform concentration inequality for scalar-valued processes that non-asymptotically matches law of the iterated logarithm and then extending this result to higher dimensions using a geometric argument. From our inequality, we can derive a multivariate analogue of the famed law of the iterated logarithm, which we show to be essentially tight. Lastly, we apply our inequality to common statistical tasks, such as calibrating confidence ellipsoids in online linear regression, estimating model parameters in vector auto-regressive models, and estimating a bounded mean via a new "empirical Bernstein" concentration inequality.

1.1 Related Work and History

Martingale concentration arguably originated in the work of Ville [56], who showed that the growth of non-negative supermartingales can be controlled uniformly over time. This result, now known commonly referred to as *Ville's inequality*, acts as a time-uniform generalization of Markov's inequality [26]. This result was later extended to submartingale concentration by Doob [24] in an eponymous result, now called *Doob's maximal inequality*. From these two inequalities, a variety of now classical martingale concentration inequalities were proved, such as Azuma's inequality [6], which serves as a time-uniform, martingale variant of Hoeffding's inequality [30] for bounded random variables, and Freedman's inequality [28], which serves as a martingale variant of Bennett's inequality [7] for sub-Poisson, bounded random variables.

Of particular note are the various self-normalized inequalities of de la Peña [22, 20, 21, 23], which provide time-uniform control of the growth of a process $(S_t)_{t\geq 0}$ in terms of an associated accumulated variance process $(V_t)_{t\geq 0}$. In particular, the authors derive their results using a technique first presented by Robbins called the method of mixtures [18, 17], which involves integrating over a family of parameterized exponential supermartingales to obtain significantly tighter (in terms of asymptotic behavior) inequalities than those mentioned earlier. Bercu and Touati [8] also investigate self-normalized concentration in the style of de la Peña, deriving bounds when the increments of $(S_t)_{t\geq 0}$ may exhibit asymmetric heavy-tailed behavior and, in later work, [9] study the effects of weighing predictable and empirical quadratic variation processes in deriving self-normalized concentration results.

Recently, Howard et al. [31] presented a single "master" theorem that ties together much of the literature surrounding scalar-valued concentration (self-normalized or not). Inspired by the classical Cramer-Chernoff method (see Boucheron et al. [11] for instance), which provides high probability tail bounds for a random variable X in terms of its cumulant generating

function (or CGF) ψ, the authors present a unified "sub-ψ" condition on a stochastic process. This condition relates the growth of a process $(S_t)_{t \geq 0}$ to some corresponding accumulated variance process $(V_t)_{t \geq 0}$ through a function ψ which obeys many similar properties to a CGF. In particular, the authors prove "line-crossing" inequalities for sub-ψ processes, giving a bound on the probability that $(S_t)_{t \geq 0}$ will ever cross a line parameterized by ψ and the accumulated variance $(V_t)_{t \geq 0}$. By strategically picking ψ and $(V_t)_{t \geq 0}$, the master theorem in Howard et al. [31] can be used to reconstruct, unify and even improve a variety of existing self-normalized concentration inequalities (such as those in the preceding paragraph), as well as to prove several new ones. Using these ideas in a followup work, Howard et al. [32] prove a time-uniform concentration inequality for scalar-valued processes whose rate non-asymptotically matches the law of the iterated logarithm (LIL) [26]. The only caveat to this result is that the concentration inequality only applies to sub-ψ processes when ψ is either the CGF of a sub-Gaussian (denoted ψ_N) or sub-Gamma (denoted $\psi_{G,c}$) random variable. While any CGF-like function ψ function can be bounded by $a\psi_{G,c}$ for *some* choice of $a, c > 0$ (see Proposition 1 of Howard et al. [32]), this conversion could in general result in loose constants. As a stepping stone toward proving our multivariate concentration inequalities, we generalize the non-asymptotic LIL results of Howard et al. [32] to arbitrary sub-ψ process, greatly increasing the applicability of the obtained results.

To the best of our knowledge, there are relatively few existing results on the self-normalized concentration of vector-valued processes. De la Peña [22] leverage the above-mentioned method of mixtures alongside Ville's inequality to bound the probability that the self-normalized random vector $V_t^{-1/2} S_t \in \mathbb{R}^d$ belongs to some mixture-dependent convex set. These bounds are, in particular, not closed form, and their asymptotic rate of growth is unclear. Our bounds, instead, directly provide time-uniform bounds on the process $(\|V_t^{-1/2} S_t\|)_{t \geq 0}$ in terms of relatively simple function of the variance process $(V_t)_{t \geq 0}$. In particular, we use our bounds to derive a multivariate law of the iterated logarithm that is tight in terms of dependence on V_t and the ambient dimension d up to small, absolute, known constants. While de la Peña et al. [21, 22] do provide an asymptotic LIL for vector processes, it hides an unknown constant and lacks explicit dependence on the dimension d.

In the case where the increments of $(S_t)_{t \geq 0}$ satisfy a sub-Gaussian condition, significantly more is known about vector-valued self-normalized concentration. Abbasi-Yadkori et al. [2] provide a clean bound on $\|V_t^{-1/2} S_t\|$ in terms of $\log \det(V_t)$ using an argument that directly follows from an earlier, method-of-mixtures based argument of de la Peña et al. [21]. First, our bounds are significantly more general than those of Abbasi-Yadkori et al. [2] and de la Peña et al. [21], because ours apply to *general* sub-ψ processes. Additionally, our bounds grow proportionally to $\log \log \gamma_{\max}(V_t)$ and $\log \kappa(V_t)$ ($\gamma_{\max}$ and κ represent maximum eigenvalue and condition number respectively, defined later). Thus, even in the setting of sub-Gaussian increments with predictable covariance, our results are not directly comparable in general. We believe deriving log-determinant rate inequalities for general sub-ψ processes is an interesting open problem, but leave it for future work.

There exist other concentration inequalities for vector-valued data that are not directly related to the self-normalized bounds presented in this paper. First, there are several existing time-uniform concentration results for Banach space-valued martingales [46, 47, 31]. These results are obtained by placing a smoothness assumption on the norm of the Banach space, and in turn provide time-uniform control on the norm of the martingale. We note

that although we are working in a Banach space, we are not trying to control the norm of the underlying process $\|S_t\|$, and instead want to control the self-normalized quantity $\left\|V_t^{-1/2} S_t\right\|$. In particular, the process $(V_t^{-1/2} S_t)_{t \geq 0}$ is not in general a martingale, so the above results cannot be directly applied. Second, there are many concentration results that involve bounding the operator norm of Hermitian matrix-valued martingales using the matrix Chernoff method [5, 13, 54, 55]. Once again, it does not seem like these bounds for matrix-valued processes can be readily applied to obtain vector-valued concentration of the form presented in this paper. Third, in their work on estimating convex divergences, Manole and Ramdas [44] derive a self-normalized concentration inequality for i.i.d. random vectors drawn from some distribution on $\mathbb{R}^d$. The form of this bound resembles that of the central concentration inequality presented in this paper. However, we note that our result allows for arbitrary martingale dependence between the increments of the process $(S_t)_{t \geq 0}$. Furthermore, the argument used in Manole and Ramdas [44] cannot be generalized to the setting of arbitrary dependence, as the authors derive their results using certain reverse martingale arguments which must be conducted with respect to the exchangeable filtration generated by a sequence of random variables, which implies the increments of $(S_t)_{t \geq 0}$ must, at the very least, be exchangeable random variables.

1.2   Our Contributions

We now provide a brief, illustrative summary of our primary contributions. For now, when we refer to a process (S_t) being sub-ψ with variance proxy (V_t), the reader should think of the increments of S_t having associated CGF ψ with weights proportional to V_t. This is not precise, but will be made exact when we provide rigorous definitions of the sub-ψ condition for both scalar and vector-valued processes in Section 2 below. We present the primary contributions in the order they appear in the paper.

1. First, in Section 3, we show that if $(S_t)_{t \geq 0}$ is a scalar (i.e. $\mathbb{R}$-valued) sub-ψ process with variance proxy $(V_t)_{t \geq 0}$, then, with high probability, it holds that

$$S_t = O\left(V_t \cdot (\psi^*)^{-1}\left(\frac{1}{V_t} \log\log(V_t)\right)\right)$$

for all $t \geq 0$ simultaneously. In the case where $\psi(\lambda) = \psi_{G,c}(\lambda) := \frac{\lambda^2}{2(1-c\lambda)}$ is the CGF associated with a sub-Gamma random variable (see Boucheron et al. [11]), our bound reduces to

$$S_t = O\left(\sqrt{V_t \log\log(V_t)} + c \log\log(V_t)\right).$$

Thus, this result can be reviewed as a direct generalization of the primary contributions of Howard et al. [32], who only provide time-uniform, self-normalized concentration results for sub-Gamma processes (Note that in the special case $c = 0$, sub-Gamma concentration reduces to sub-Gaussian concentration).

2. Next, in Section 4, we show that if $(S_t)_{t \geq 0}$ is a vector valued process that is sub-ψ with variance proxy $(V_t)_{t \geq 0}$, then, with high probability, simultaneously for all $t \geq 0$,

$$\|V_t^{-1/2} S_t\| = O\left(\sqrt{\gamma_{\min}(V_t)} \cdot (\psi^*)^{-1}\left(\frac{1}{\gamma_{\min}(V_t)}\left[\log\log(\gamma_{\max}(V_t)) + d\log\kappa(V_t)\right]\right)\right),$$

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where $\gamma_{\min}(V_t)$ and $\gamma_{\max}(V_t)$ refer, respectively, to the minimum and maximum eigenvalues of the (proxy) covariance matrix V_t, and $\kappa(V_t) := \frac{\gamma_{\max}(V_t)}{\gamma_{\min}(V_t)}$ is the condition number of matrix V_t. We can compare our bounds to existing results [21, 2] in the subGaussian case where $\psi(\lambda) = \psi_N(\lambda) := \frac{\lambda^2}{2}$ is the CGF of a standard normal random variable. In this case, our bound simplifies to

$$\|V_t^{-1/2} S_t\| = O\left(\sqrt{\log\log(\gamma_{\max}(V_t)) + d\log\kappa(V_t)}\right).$$

Existing results on the self-normalized concentration for processes with sub-Gaussian increments provide upper bounds of the form $\|V_t^{-1/2} S_t\| = O\left(\sqrt{\log\det(V_t)}\right)$, which are, in general, incomparable to our bounds. When $\kappa(V_t)$ is small, our bounds may be tighter, but if $\gamma_{\max}(V_t) \gg \gamma_{\min}(V_t)$, the determinant-based bounds may be tighter.

3. Lastly, in Sections 5 and 6, we apply our vector-valued self-normalized concentration results to statistical tasks. In Section 5, we create non-asymptotically valid confidence ellipsoids for estimating unknown slope parameters in online linear regression with sub-ψ noise in observations. In particular, these results can be viewed as extending the confidence ellipsoids of Abbasi-Yadkori et al. [2], which hold only in the sub-Gaussian setting. We further specialize these bounds to model estimation in vector autoregressive models (i.e. in the VAR(p) model), generalizing a result of Bercu and Touati [8]. In Section 6, we prove a multivariate, self-normalized empirical Bernstein inequality, generalizing a result of Howard et al. [32] to d-dimensional space.

In sum, we provide time-uniform, self-normalized concentration inequalities for both scalar and vector-valued processes that hold under quite general noise conditions. Not only are these bounds of theoretical interest, but they are in fact applicable to common statistical tasks — in particular those that can be framed in the online linear regression framework.

2 Background and Sub-ψ Processes

In this section we discuss the key sub-ψ condition leveraged in deriving self-normalized concentration results for vector-valued processes. We arrive at our vector sub-ψ condition by extending the eponymous condition defined in the setting of scalar-valued processes [31, 32], to high dimensional spaces. We first summarize some notation that will be used ubiquitously.

Notation: Throughout, we define $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$ to be the set of natural numbers, which we assume to begin at 0. We let $\langle x, y \rangle = x^\top y$ denote that standard Euclidean inner product on $\mathbb{R}^d$. Additionally, we let $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$ denote the unit sphere and $\mathbb{B}_d := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$ the unit ball in $\mathbb{R}^d$. By $\mathcal{L}_+(\mathbb{R}^d)$, we denote the set of all $d \times d$ positive semi-definite matrices, with $I_d \in \mathcal{L}_+(\mathbb{R}^d)$ denoting the d-dimensional identity matrix. For $V \in \mathcal{L}_+(\mathbb{R}^d)$, let $\gamma_{\max}(V)$ denote the largest eigenvalue of V, $\gamma_{\min}(V)$ the smallest eigenvalue of V, and let

$$\kappa(V) := \frac{\gamma_{\max}(V)}{\gamma_{\min}(V)}$$

denote the condition number of V. Each such V admits a spectral decomposition of the form $V = \sum_{n=1}^{d} \gamma_n(V) v_n v_n^\top$, where $(\gamma_n(V))_{n \in [d]}$ is the non-increasing sequence of eigenvalues associated with matrix V and $(v_n)_{n \in [d]}$ is the corresponding sequence of unit eigenvectors, which we know forms an orthonormal basis for $\mathbb{R}^d$. For $\rho > 0$, let

$$V \vee \rho I_d := \sum_{n=1}^{d} (\gamma_n(V) \vee \rho) v_n v_n^\top,$$

where for scalars $a, b \in \mathbb{R}$, $a \vee b := \max\{a, b\}$.

For a strictly increasing, differentiable convex function $\psi : [0, \lambda_{\max}) \to \mathbb{R}$ we let $\psi^* : [0, u_{max}) \to \mathbb{R}_{\geq 0}$ denote its convex conjugate, given by $\psi^*(u) := \sup_{\lambda \in [0, \lambda_{\max})} u\lambda - \psi(\lambda)$, where $u_{\max} := \lim_{\lambda \uparrow \lambda_{\max}} \psi'(\lambda)$. In the sequel, we will always assume $\sup_{\lambda \in [0, \lambda_{\max})} \psi'(\lambda) = \infty$, and hence will have $u_{\max} = \infty$. Some key properties of convex conjugation are that (a) ψ^* is convex, (b) $(\psi^*)^* = \psi$, and (c) $(\psi^*)' = (\psi')^{-1}$.

Let (Z, ρ) denote a metric space, and let $T \subset Z$. For $\epsilon > 0$, we say that a set $K \subset Z$ is an ϵ-covering for T if, for any $z \in T$, there exists a point $\pi(z) \in K$ satisfying $\rho(z, \pi(z)) \leq \epsilon$. We call $\pi : T \to K$ a "projection" onto the covering, which maps each point in T onto the nearest point in K (or an arbitrary one if not unique). If $K \subset T$, we call K a *proper* ϵ-covering of T. We will exclusively consider proper coverings in the sequel. We define the ϵ-covering number $N(T, \epsilon, \rho)$ of T to be the cardinality of the smallest proper ϵ-covering of T. Any proper ϵ-covering of T obtaining this minimum will be called minimal. In the special case $(Z, \rho) = (\mathbb{R}^d, \|\cdot\|)$ and $T = \mathbb{S}^{d-1}$, we denote the ϵ-covering number of T by $N_{d-1}(\epsilon)$.

Lastly, if $(S_t)_{t \geq 0}$ is some process evolving in a space $\mathcal{X}$ and $t \geq 1$, we define the tth increment of $(S_t)_{t \geq 0}$ to be $\Delta S_t := S_t - S_{t-1}$. If a filtration $(\mathcal{F}_t)_{t \geq 0}$ is understood from context, we may use the notation $\mathbb{E}_t[\cdot] = \mathbb{E}(\cdot \mid \mathcal{F}_t)$ for easing notational burden. By default, we take $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_t = \sigma(S_1, \ldots, S_t)$.

Sub-ψ Processes: We now describe in more detail a condition that links the growth of a process $(S_t)_{t \geq 0}$ evolving in $\mathbb{R}^d$ to a corresponding "accumulated variance process" $(V_t)_{t \geq 0}$ taking values in $\mathcal{L}_+(\mathbb{R}^d)$. This linking will occur through the consideration of a family of exponential processes in which a scaled version of $(S_t)_{t \geq 0}$ along any fixed direction is compensated by $(V_t)_{t \geq 0}$ and a function ψ that measures the heaviness of the tails of ΔS_t. ψ should be thought of as acting like the cumulant generating function (or CGF) of ΔS_t — we will make this notion precise in our later discussion of CGF-like functions. These exponential processes will behave like non-negative supermartingales, and thus will allow us to apply powerful time-uniform concentration results to bound the growth of an appropriately normalized version of $(S_t)_{t \geq 0}$. Due to the central of role ψ in connecting the growth of $(S_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$, we will adopt the terminology of Howard et al. [31, 32] from the scalar case and refer to the condition as the "sub-ψ condition".

Before formally defining the sub-ψ condition, we must briefly discuss the properties of the heretofore vaguely defined function ψ. ψ will be a cumulant generating function-like (or CGF-like) function, which roughly means it behaves like the CGF of some random variable. More explicitly, by a CGF-like function, we mean a twice continuously-differentiable function $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ satisfying (a) ψ is strictly convex, (b) $\psi(0) = \psi'(0) = 0$, and (c)

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$\psi''(0) > 0$. Notable examples of CGF-like functions include $\psi_N(\lambda) := \frac{\lambda^2}{2}$, the CGF of a standard normal random variable;

$$\psi_{E,c}(\lambda) := \frac{-\log(1 - c\lambda) - c\lambda}{c^2},$$

the CGF of a (centered) exponential random variable; $\psi_{P,c}(\lambda) := \frac{e^{c\lambda} - c\lambda - 1}{c^2}$, the CGF of a centered Poisson random variable; and

$$\psi_{G,c}(\lambda) := \frac{\lambda^2}{2(1 - c\lambda)},$$

a bound on the CGF of a centered Gamma random variable. Note that, in particular, $\psi_{G,0} = \psi_N$. Basic theory regarding CGF-like functions is discussed in detail in Appendix A. While we will use many nontrivial properties of CGF-like functions freely hereinafter, we will always make the proper forward reference to Appendix A.

We now present the sub-ψ condition for scalar processes, and later for vector processes. First introduced in Howard et al. [31], the sub-ψ condition very heuristically states that, for each $t \geq 0$, the cumulant generating function for S_t is dominated by $V_t \cdot \psi$, where ψ is some CGF-like function per the above definition. More precisely, the sub-ψ condition for scalar-valued processes is as follows.

Definition 2.1. Let $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ be CGF-like, let $(S_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ be respectively $\mathbb{R}$-valued and $\mathbb{R}_{\geq 0}$-valued processes adapted to some filtration $(\mathcal{F}_t)_{t \geq 0}$. We say that $(S_t, V_t)_{t \geq 0}$ is sub-ψ (or equivalently that $(S_t)_{t \geq 0}$ is a sub-ψ process with variance proxy $(V_t)_{t \geq 0}$) if for every $\lambda \in [0, \lambda_{\max})$, the exponential process $\exp\{\lambda S_t - \psi(\lambda) V_t\}$ is (almost surely) upper bounded by some non-negative supermartingale $(L_t^\lambda)_{t \geq 0}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$:

$$M_t^\lambda := \exp\{\lambda S_t - \psi(\lambda) V_t\} \leq L_t^\lambda, \qquad \text{for all } t \geq 0.$$

As an easy example, consider the case where $(X_t)_{t \geq 1}$ is a sequence of i.i.d. mean zero random variables with CGF $\psi(\lambda) = \log \mathbb{E} e^{\lambda X_1}$. Letting $S_t := \sum_{s=1}^t X_s$ and $V_t := t$, it is easy to see that M_t^λ is a non-negative martingale with respect to the natural filtration generated by the X_t's (and thus we can take $L_t^\lambda = M_t^\lambda$).

Definition 2.1 generalizes the above example to a setting where the random variables may have more complicated dependence structures, and "nonparametric" tail conditions, including settings where V_t can itself be adapted to $(\mathcal{F}_t)_{t \geq 0}$ (as opposed to constant or predictable variance processes), a key ingredient in self-normalized bounds. Recently, Howard et al. [31] compiled a rich selection of examples of such sub-ψ processes. For more examples, one can specialize each of the multivariate sub-ψ processes following Definition 2.2 below to the setting $d = 1$.

The above definition for scalar-valued processes suggests a straightforward means of generalizing the sub-ψ condition to the setting where $(S_t)_{t \geq 0}$ is $\mathbb{R}^d$-valued and $(V_t)_{t \geq 0}$ is $\mathcal{L}_+(\mathbb{R}^d)$-valued. Namely, $(S_t, V_t)_{t \geq 0}$ should be sub-ψ if, for any direction $\nu \in \mathbb{S}^{d-1}$, the scalar-valued process $(\langle \nu, S_t \rangle, \langle \nu, V_t \nu \rangle)_{t \geq 0}$ is sub-ψ. We formalize this in the following definition, which recovers Definition 2.1 in the case $d = 1$.

Definition 2.2. Let $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ be CGF-like, and let $(S_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ be respectively $\mathbb{R}^d$-valued and $\mathcal{L}_+(\mathbb{R}^d)$-valued processes adapted to some filtration $(\mathcal{F}_t)_{t \geq 0}$. We say that $(S_t, V_t)_{t \geq 0}$ is sub-ψ if, for every $\nu \in \mathbb{S}^{d-1}$, the projected process $(\langle \nu, S_t \rangle, \langle \nu, V_t \nu \rangle)_{t \geq 0}$ is sub-ψ in the sense of Definition 2.1. In other words, $(S_t, V_t)_{t \geq 0}$ is sub-ψ if, for any $\nu \in \mathbb{S}^{d-1}$ and $\lambda \in [0, \lambda_{\max})$, there is a non-negative supermartingale $(L_t^{\lambda \cdot \nu})_{t \geq 0}$ with respect to $(\mathcal{F}_t)_{t \geq 0}$ such that

$$M_t^{\lambda \cdot \nu} := \exp \left\{ \lambda \langle \nu, S_t \rangle - \psi(\lambda) \langle \nu, V_t \nu \rangle \right\} \leq L_t^{\lambda \cdot \nu}, \qquad \text{for all } t \geq 0.$$

It is straightforward to confirm that if $(S_t, V_t)_{t \geq 0}$ is sub-ψ, then $(S_t, V_t + \rho I_d)_{t \geq 0}$ and $(S_t, V_t \vee \rho I_d)_{t \geq 0}$ are sub-ψ as well. Furthermore, it is also straightforward to check that the rescaled process $(S_t/\sqrt{\rho}, V_t/\rho)_{t \geq 0}$ is sub-ψ_ρ, where $\psi_\rho : [0, \sqrt{\rho}\lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is given by

$$\psi_\rho(\lambda) := \rho \psi(\lambda/\sqrt{\rho}).$$

These transformations are important as they will allow us to exclusively study processes satisfying $V_1 \geq 1$ in the sequel. For the sake of completeness, we prove that ψ_ρ is in fact CGF-like in Proposition A.2 in Appendix A. We codify the above observations into the following proposition for ease of reference.

Proposition 2.3. *Suppose $(S_t, V_t)_{t \geq 0}$ is sub-ψ with (inherently with respect to some filtration $(\mathcal{F}_t)_{t \geq 0}$). Then, for any fixed $\rho > 0$,*

1. *$(S_t, V_t + \rho I_d)_{t \geq 0}$ is sub-ψ with respect to $(\mathcal{F}_t)_{t \geq 0}$,*

2. *$(S_t, V_t \vee \rho I_d)_{t \geq 0}$ is sub-ψ with respect to $(\mathcal{F}_t)_{t \geq 0}$, and*

3. *$(S_t/\sqrt{\rho}, \rho^{-1} V_t)_{t \geq 0}$ is sub-ψ_ρ with respect to $(\mathcal{F}_t)_{t \geq 0}$, where $\psi_\rho(\lambda) := \rho \psi(\lambda/\sqrt{\rho})$.*

As we will see, Definition 2.2 will prove to be the "right" generalization of the sub-ψ condition to high-dimensional settings. In more detail, from the condition, we will derive a general, time-uniform bound on the self-normalized process $(\|V_t^{-1/2} S_t\|)_{t \geq 0}$ that will be tight up to small, multiplicative constants.

Four Examples of Sub-ψ Processes: We now provide four practically-relevant examples of multivariate sub-ψ processes — one for each of the aforementioned CGF-like functions $\psi_N, \psi_P, \psi_{E,c}$, and $\psi_{G,c}$. In each of the examples below, we assume we are studying some process $(X_t)_{t \geq 1}$ that is adapted to some filtration $(\mathcal{F}_t)_{t \geq 0}$.

1. If $X_t =_d -X_t \mid \mathcal{F}_{t-1}$ (that is, the X_t are conditionally symmetric), Lemma 3 of de la Peña et al. [21] can be used to show that $S_t := \sum_{s=1}^t X_s$ is sub-ψ_N with variance proxy $V_t := \sum_{s=1}^t X_s X_s^\top$. This provides salient example of how the sub-ψ condition can be leveraged to provide meaningful concentration for processes whose increments may even lack a well-defined mean (e.g. take the X_t to be i.i.d. Cauchy random variables).

2. If $\|X_t\| \leq c$ almost surely, a standard Bennett-style argument (see the proof of Theorem 2.9 in Boucheron et al. [11]) shows that $S_t := \sum_{s=1}^t \left\{ X_s - \mathbb{E}_{s-1} X_s \right\}$ is sub-$\psi_{P,c}$ with variance proxy $V_t := \sum_{s=1}^t \mathbb{E}_{s-1} X_s X_s^\top$.

3. As will be seen in Section 6, if $\|X_t\| \leq 1/2$ almost surely[1], then $S_t := \sum_{s=1}^t \{X_s - \mathbb{E}_{s-1}X_s\}$ is sub-$\psi_{E,1}$ with variance proxy $V_t := \sum_{s=1}^t (X_s - \widehat{\mu}_{s-1})(X_s - \widehat{\mu}_{s-1})^\top$. In the above, $\widehat{\mu}_t := \frac{1}{t}\sum_{s=1}^t X_s$ is the time-average mean given the first t samples. From this condition, one can derive a multivariate, self-normalized "empirical Bernstein" inequality. This type of inequality is useful in statistical applications [58] due to the fact its tightness adapts to the observed (i.e. empirical) variance within the samples witnessed.

4. Lastly, if $\mathbb{E}_{t-1}|\langle \nu, X_t \rangle|^k \leq \frac{k!}{2}c^{k-2}\mathbb{E}_{t-1}\langle \nu, X_t \rangle^2$ for all directions $\nu \in \mathbb{S}^{d-1}$ and some constant $c > 0$, a standard application of the Bernstein condition in each direction $\nu \in \mathbb{S}^{d-1}$ (see Theorem 2.10 of Boucheron et al. [11]) yields that $S_t := \sum_{s=1}^t \{X_s - \mathbb{E}_{s-1}X_s\}$ is sub-$\psi_{G,c}$ with variance proxy $V_t := \sum_{s=1}^t \mathbb{E}_{s-1}X_sX_s^\top$.

Super-Gaussian CGFs: We draw attention to *super-Gaussian* ψ:

a CGF-like function ψ is super-Gaussian if $\frac{\psi(\lambda)}{\lambda^2}$ is an increasing function of λ.

In words, ψ is super-Gaussian if it grows at least as rapidly as ψ_N, the CGF of a $\mathcal{N}(0,1)$ random variable. Most notable examples of CGF-like functions are super-Gaussian, with particularly important examples being $\psi_N, \psi_{E,c}, \psi_{G,c}$, and $\psi_{P,c}$. Informally, one typically needs to use a super-Gaussian CGF if the underlying random process is heavier tailed than a sub-Gaussian process.

One example of a CGF that is not super-Gaussian would be $\psi_{B,p}(\lambda)$, the CGF of a centered Bernoulli random variable X with $\mathbb{P}(X = 1) = p$. We discuss equivalent definitions and properties of CGF-like functions in detail in Appendix A. While our bounds will hold in the case where $(S_t, V_t)_{t\geq 0}$ is sub-ψ for arbitrary ψ, they are particularly clean when ψ is super-Gaussian, and we emphasize this case going forward.

3 A General Non-Asymptotic LIL for Scalar Processes

In this section, we prove a high-probability, time-uniform bound on the growth of a scalar process $(S_t)_{t\geq 0}$ normalized by some measure of accumulated variance $(V_t)_{t\geq 0}$. In particular, in Theorem 3.1 below, we show that if $(S_t, V_t)_{t\geq 0}$ is a sub-ψ process, then, with high probability, simultaneously for all $t \geq 0$,

$$S_t = O\left(V_t \cdot (\psi^*)^{-1}\left(\frac{1}{V_t}\log\log(V_t)\right)\right),$$

where we have omitted dependence on several user-chosen parameters and constants for the sake of exposition. Dividing both sides by $\sqrt{V_t}$ yields a result in "self-normalized" form that looks more akin to the results in subsequent sections, but we adopt the above form for consistency with existing results [31, 32]. Since $(\psi^*)^{-1}(u) \sim \sqrt{2u}$ as $u \downarrow 0$ whenever $\psi(\lambda) \sim \frac{\lambda^2}{2}$ as $\lambda \downarrow 0$ (as is the case for all CGF-like functions addressed in the previous section), for large values of V_t, the above high probability bound can be written as

$$S_t = O(\sqrt{V_t \log\log(V_t)}),$$

---

[1]Note that the the assumption $\|X_t\| \leq \frac{1}{2}$ can be replaced by any constant by appropriately changing the scale parameter of the sub-Exponential CGF.

thus allowing our results in this section to be viewed as a non-asymptotic (i.e. finite sample) version of the law of the iterated logarithm. We further describe connections between our scalar-valued bound and the law of the iterated logarithm in Subsection 3.2 below.

While we construct the bounds in this section as a requisite for deriving self-normalized concentration inequalities for vector-valued processes, we believe the results are of independent interest. In particular, our results are significantly more general than those of Howard et al. [32], whose bounds serve as the current state-of-the-art for scalar-valued self-normalized concentration. Unlike the results of Howard et al. [32], which only hold for sub-$\psi_{G,c}$ (i.e. sub-Gamma) processes, our results hold for *general* sub-ψ processes. While Howard et al. [31] show that any CGF-like function ψ can be bounded pointwise by $a\psi_{G,c}$ for appropriately chosen constants $a, c > 0$, this comparison can be loose. We illustrate this in Figure 1 in Appendix D, which shows that the time-uniform boundary presented in Theorem 3.1 (applied in the sub-Poisson setting $\psi = \psi_{P,c}$) can offer improved concentration over the main theorem of Howard et al. [32], which requires converting sub-$\psi_{P,c}$ process to sub-$\psi_{G,c}$ processes in order to be applied. This case is of particular interest, as straightforward calculation shows $\psi_{P,c}(\lambda) \leq \psi_{G,c}(\lambda)$ for all λ. Other examples demonstrating this disparity in bounds can be readily constructed as well. Furthermore, even in the case of sub-$\psi_{G,c}$ processes, our bounds are essentially flush with those of Howard et al. [32] in the sub-Gamma case, being multiplicatively looser by a vanishingly small factor as certain tuning parameters are appropriately selected, as illustrated in Figure 2, also in Appendix D. We further discuss comparisons between our bounds and those of Howard et al. [32] following the proof of Theorem 3.1.

Before presenting the main theorem of this section, we discuss heuristically how we are able to generalize the results of Howard et al. [32]. Much like the "stitching" technique of the aforementioned authors, our argument proceeds, roughly, by breaking "intrinsic" time into geometric epochs of the form $\{\alpha^k \leq V_t < \alpha^{k+1}\}$ and then optimizing a tight linear inequality in each period. The key difference in our argument is in how we optimize this linear boundary for $(S_t)_{t \geq 0}$ in each epoch. The techniques leveraged by Howard et al. [32] yield a boundary that is defined in terms of $\psi_{G,c}^{-1}$, the inverse of the CGF-like function associated with a Gamma distribution. From our understanding of the classical Chernoff argument, we know that if a mean zero random variable X has associated CGF $\psi(\lambda) := \log \mathbb{E}e^{\lambda X}$, then we have $\mathbb{P}\left(X \geq (\psi^*)^{-1}\left(\log\left(\frac{1}{\delta}\right)\right)\right) \leq \delta$. Thus, although the Chernoff argument doesn't directly apply in this time-uniform setting, we at the very least expect to obtain a boundary defined in terms of $(\psi^*)^{-1}$. By coupling this intuition with the time-uniform line crossing inequalities of Howard et al. [31], we are able to obtain an extremely general inequality for sub-ψ processes with a surprisingly straightforward argument.

With the above discussion in mind, we present Theorem 3.1.

Theorem 3.1. *Suppose $(S_t, V_t)_{t \geq 0}$ is a real-valued sub-ψ process for some CGF-like function $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ satisfying $\lim_{\lambda \uparrow \lambda_{\max}} \psi'(\lambda) = \infty$. Let $\alpha > 1, \rho > 0$, and $\delta \in (0, 1)$ be constants respectively representing the stitching epoch length, the minimum intrinsic time, and the error probability. Let $h : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ be an increasing function such that $\sum_{k \in \mathbb{N}_0} h(k)^{-1} \leq 1$, representing how the error is spent across epochs. Define the function $\ell_\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by*

$$\ell_\rho(v) = \log\left(h\left(\log_\alpha\left(\frac{v \vee \rho}{\rho}\right)\right)\right) + \log\left(\frac{1}{\delta}\right),$$

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where we have suppressed the dependence of $\ell_\rho(v)$ on α, h for brevity. Then, we have

$$\mathbb{P}\left(\exists t \geq 0 : S_t \geq (V_t \vee \rho) \cdot (\psi^*)^{-1}\left(\frac{\alpha}{V_t \vee \rho}\ell_\rho(V_t)\right)\right) \leq \delta.$$

We provide a full proof of Theorem 3.1 in Section 7 below. Except for the unavoidable error probability δ, we briefly elaborate on the other user-specified constants that appear in the statement of the theorem:

1. $\alpha > 1$ controls the spacing of "intrinsic time" or accumulated variance of the process $(S_t)_{t \geq 0}$. Heuristically, Theorem 3.1 will be obtained by optimizing tight, linear boundaries on events of the form $\{\alpha^k \leq V_t < \alpha^{k+1}\}$.

2. $\rho > 0$ gives the first "intrinsic time" at which our boundaries start depending on the variance process $(V_t)_{t \geq 0}$. When $0 \leq V_t < \rho$, the boundary will only depend on ρ.

3. $h : \mathbb{N}_0 \to \mathbb{R}_{>0}$ is a function satisfying $\sum_{k \geq 0} h(k)^{-1} \leq 1$. h defines how much of the overall probability mass associated with failure (determined by δ) to allocate to each event of the form $\{\alpha^k \leq V_t < \alpha^{k+1}\}$.

In the above, we view the parameters ρ, and h as critical, since they directly affect the shape and validity of the bound, whereas we view α as less critical, as any small variation in α will only minimally affect the tightness of the bound in terms of constants. For example, a reasonable choice of this temporal spacing parameter is $\alpha = 1.05$. Howard et al. [32] discuss reasonable choices for the function h, and we emphasize in the sequel the choice of $h(k) := (k+1)^s \zeta(s)$, where $s > 1$ is a tuning parameter and ζ is the Riemann zeta function. This choice is of particular theoretical interest as it yields non-asymptotic rates that depend on $\log\log(V_t)$ (up to constants), thus allowing our bound to be viewed as a general, non-asymptotic version of the LIL. We in particular use this choice of h in the proof of Corollary 3.2 in Subsection 3.2 below.

3.1 Comparison With Existing Bounds

We compare our results to those presented in Theorem 1 of Howard et al. [32], who provide time-uniform, self-normalized concentration for scalar processes in the sub-$\psi_{G,c}$ case, where we recall $\psi_{G,c}(\lambda) := \frac{\lambda^2}{2(1-c\lambda)}$ is the CGF-like function associated with a sub-Gamma random variable. We start by analyzing the special case $c = 0$, in which $\psi_{G,0} = \psi_N$ is the CGF of a $\mathcal{N}(0, 1)$ random variable. In our notation, the authors show that if $(S_t, V_t)_{t \geq 0}$ is a sub-ψ_N process, then, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$,

$$S_t \leq \sqrt{\left(\frac{\alpha^{1/4} + \alpha^{-1/4}}{\sqrt{2}}\right)^2 (V_t \vee \rho)\ell_\rho(V_t)}.$$

Noting that $(\psi_N^*)^{-1}(u) = \sqrt{2u}$, our results yield that, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$,

$$S_t \leq \sqrt{2\alpha(V_t \vee \rho)\ell_\rho(V_t)}.$$

A straightforward computation yields that, for all $\alpha > 1$, $\left(\frac{\alpha^{1/4}+\alpha^{-1/4}}{\sqrt{2}}\right)^2 \leq 2\alpha$, showing that the bounds of Howard et al. [32] are (slightly) tighter than our own. However, for $\alpha < 2.06$, we have $2\alpha \leq 2\left(\frac{\alpha^{1/4}+\alpha^{-1/4}}{\sqrt{2}}\right)^2$, showing that our bounds are looser than those of Howard et al. [32] by a multiplicative factor of no more than $\sqrt{2}$ in this regime. In particular, as α is decreased towards 1, the multiplicative factor by which our bounds are suboptimal to those of Howard et al. [32] vanishes to 1.

In the general case of $c > 0$, Theorem 1 in Howard et al. [32] yields that with probability at least $1 - \delta$, we have

$$S_t \leq \sqrt{\left(\frac{\alpha^{1/4}+\alpha^{-1/4}}{\sqrt{2}}\right)^2 (V_t \vee \rho)\ell_\rho(V_t) + \left(\frac{\sqrt{\alpha}+1}{2}\right)^2 \ell_\rho(V_t)} + c\left(\frac{\sqrt{\alpha}+1}{2}\right)\ell_\rho(V_t). \quad (1)$$

Meanwhile, noting that $(\psi_{G,c}^*)^{-1}(x) = \sqrt{2x}+cx$ (this can be readily checked, or see Boucheron et al. [11] e.g.), our results yield that, with probability at least $1 - \delta$, we have

$$S_t \leq \sqrt{2\alpha(V_t \vee \rho)\ell_\rho(V_t)} + c\alpha\ell_\rho(V_t).$$

In this situation, a general comparison between the resulting bounds isn't clear. Our second term is larger, but it is a lower order term. Regarding the first term, for small V_t and α, we may expect our bound to be tighter, as we don't suffer from the second additive term inside of the square root. On the other hand, for any $\alpha > 1$ and moderate to large V_t, by our analysis in the sub-Gaussian case of $\psi = \psi_N$ (which is recovered when $c = 0$), we expect the bound from Howard et al. [32] to be tighter for the same reason. We plot a detailed comparison between Theorem 3.1 applied in the sub-$\psi_{G,c}$ case and Equation 1 in Figure 2, found in Appendix D.

We emphasize that the bounds of Howard et al. [32] hold *only* in the sub-Gamma case. While sub-Gamma concentration can be applied to sums of sub-Exponential and sub-Poisson random variables, this approximation is far from tight, especially in small sample sizes. Our results hold directly for *any* CGF-like function ψ, including all listed in Section 2.

3.2   Asymptotic Law of the Iterated Logarithm

In the preceding paragraphs, we derived time-uniform bounds for general scalar-valued sub-ψ processes. In particular, we argued our presented results generalized those of Howard et al. [32], who show a similar result for the case $\psi = \psi_{G,c} = \frac{\lambda^2}{2(1-c\lambda)}$ (i.e. when ψ is the CGF-like function associated with a sub-Gamma random variable). As noted above, for any fixed step size $\alpha > 1$, in the case $c = 0$ (i.e. when $\psi = \psi_N$ is the CGF of a standard Gaussian random variable), the bounds of Howard et al. [32] dominate ours, albeit by a vanishingly small multiplicative factor as $\alpha \downarrow 1$.

This begs the following question: are our bounds "optimal" in the sense that, by appropriately selecting the tuning parameters, they recover the asymptotic (upper) law of the iterated logarithm with the correct constant. In Corollary 3.2 below, we show that this exactly the case, and thus derive a law of the iterated logarithm for sub-ψ processes.

Corollary 3.2. *Let $(S_t)_{t\geq 0}$ be sub-ψ with variance proxy $(V_t)_{t\geq 0}$, and suppose that $\psi(\lambda) \sim \frac{\lambda^2}{2}$ as $\lambda \downarrow 0$ and $V_t \xrightarrow[t\to\infty]{} \infty$. Then,*

$$\limsup_{t\to\infty} \frac{S_t}{\sqrt{2V_t \log\log(V_t)}} \leq 1 \;\; almost\; surely.$$

Corollary 3.2 follows as a direct consequence of Corollary 4.6, which provides an asymptotic law of the iterated logarithm for vector-valued sub-ψ processes, noting that the dependence of the bound on the condition number of V_t vanishes in the scalar case.

We provide some brief intuition for our proof of Corollary 3.2. In the proof, we consider a sequence of bounds (indexed by $n \geq 1$), with tuning parameters $(\alpha_n)_{n\geq 1}$, $(\delta_n)_{n\geq 1}$, and $(h_n)_{n\geq 1}$ satisfying $\alpha_n \downarrow 1$, $\delta_n \downarrow 0$, and $h_n(k) := (k+1)^{s_n}\zeta(s_n)$, where $s_n \downarrow 1$. We assume $\rho := 1$, as by assumption the variance $(V_t)_{t\geq 0}$ will grow towards infinity and thus the initial time at which the bound is valid will not matter. Heuristically, smaller values of the aforementioned parameters (being α_n, δ_n, s_n) imply that the ratio between our bounds and $\sqrt{2V_t \log\log(V_t)}$ will be closer to 1 for large values of V_t, but will suffer from an increased "bias" or additive penalty for small values of V_t. Since we assume V_t grows towards infinity almost surely, the effect of the additive bias becomes negligible in the large intrinsic time limit (i.e. as $V_t \to \infty$).

In more detail, for any $n \geq 1$, we show that for large values of time $t \geq T_n$ (T_n may be random), we have $V_t \cdot (\psi^*)^{-1}\left(\frac{\alpha}{V_t}\ell_1(V_t)\right) \leq (1+\eta)C_n\sqrt{2V_t\log\log(V_t)}$, where $C_n \downarrow 1$ as $n \to \infty$ and $\eta > 0$ is some pre-fixed parameter. Given we have such a bound for all $n \geq 1$, we can apply the first Borel-Cantelli lemma (which describes when certain sequences of events will happen either "infinitely often" or only "finitely often") to show that $\limsup_{t\to\infty}\frac{S_t}{\sqrt{2V_t\log\log(V_t)}} \leq (1+\eta)$. Since $\eta > 0$ was arbitrary, the desired bound follows.

4   Main result

We now present the main result of this paper: a time-uniform, self-normalized concentration inequality for a general class of processes evolving in $\mathbb{R}^d$. We now discuss the intuition for our argument. Our results follow by coupling our scalar-valued self-normalized inequalities, presented in the previous section, with a simple but careful geometric covering argument. At a high level, our results in the previous section could be seen as controlling the growth of the process $(S_t)_{t\geq 0}$ over various scales of "intrinsic time", determined by the accumulated variance process $(V_t)_{t\geq 0}$. Analogously, to handle the multivariate nature of results in this section, we need to carefully control how the accumulated variance process (this time a matrix-valued process), distorts the geometry of $\mathbb{R}^d$ across various scales. In this setting, the level of distortion is controlled by $\kappa(V_t) := \gamma_{\max}(V_t)/\gamma_{\min}(V_t)$, the condition number of the positive semi-definite matrix V_t (if $\gamma_{\min}(V_t) = 0$, $\kappa(V_t) = \infty$ by convention).

Theorem 4.1. *Suppose $(S_t)_{t\geq 0}$ is a sub-ψ process with variance proxy $(V_t)_{t\geq 0}$ taking values $\mathbb{R}^d$. Let $\alpha > 1$, $\beta > 1$, $\rho > 0$, $\epsilon \in (0,1)$, and $\delta \in (0,1)$ be constants, and let $h : \mathbb{N_0} \to \mathbb{R}_{\geq 0}$ be*

an increasing function such that $\sum_{k\in\mathbb{N}_0} h(k)^{-1} \leq 1$. Define the function[2] $L_\rho : \mathcal{S}_+^d \to \mathbb{R}_{\geq 0}$ by

$$L_\rho(V) := \log\left(h\left(\log_\alpha\left(\frac{\gamma_{\max}(V \vee \rho I_d)}{\rho}\right)\right)\right) + \log\left(\frac{1}{\delta}\frac{1}{1-\beta^{-1}}\right)$$

$$+ \log\left(\beta\sqrt{\kappa(V \vee \rho I_d)} \cdot N_{d-1}\left(\frac{\epsilon}{\beta\sqrt{\kappa(V_t \vee \rho I_d)}}\right)\right).$$

If ψ is super-Gaussian, meaning $\psi(\lambda)/(\lambda^2/2)$ is an increasing function of λ, then

$$\mathbb{P}\left(\exists t \geq 0 : \left\|(V_t \vee \rho I_d)^{-1/2}S_t\right\| \geq \frac{\sqrt{\gamma_{\min}(V_t \vee \rho I_d)}}{1-\epsilon} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\gamma_{\min}(V_t \vee \rho I_d)}L_\rho(V_t)\right)\right) \leq \delta.$$

In addition to the parameters α, ρ, and h from Theorem 3.1, there are two new user-specified constants that govern the geometric aspects of our bound presented in Theorem 4.1.

1. $\beta > 1$ controls the spacing of how the action of the sequence of matrices $(V_t)_{t\geq 0}$ distorts the geometry of $\mathbb{R}^d$. Heuristically, Theorem 4.1 will be obtained by optimizing self-normalized inequalities on events of the form $\{\beta^k \leq \sqrt{\kappa(V_t)} < \beta^{k+1}\}$ and carefully performing a union bound.

2. $\epsilon \in (0,1)$ controls the "mesh" or level of granularity at which we approximate the geometry of the unit sphere $\mathbb{S}^{d-1}$ in the covering argument we make.

In the vocabulary of our preceding results, we view neither β nor ϵ as being critical parameters in optimizing our boundary. In particular, for simplicity, reasonable default choices would be $\beta = 2$ and $\epsilon = \frac{1}{2}$.

Before proving Theorem 4.1, we comment that a result similar to the above holds even in the setting where the CGF-like function ψ is not super-Gaussian. In particular, en route to proving the above, we will show that, if $(S_t, V_t)_{t\geq 0}$ is sub-ψ for any CGF-like ψ, we have

$$\mathbb{P}\left(\exists t \geq 0 : \left\|(V_t \vee \rho I_d)^{-1/2}S_t\right\| \geq \sup_{\nu\in\mathbb{S}^{d-1}} \frac{\sqrt{\langle \nu, V_t\nu\rangle}}{1-\epsilon} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\langle \nu, V_t\nu\rangle}L_\rho(V_t)\right)\right) \leq \delta.$$

The assumption that ψ is super-Gaussian merely allows us to compute the supremum over $\nu \in \mathbb{S}^{d-1}$ in the above expression, giving the result a cleaner form. This assumption is not restrictive, as many reasonable examples of CGF-like functions are super-Gaussian (e.g. ψ_N, $\psi_{G,c}$, $\psi_{E,c}$, and $\psi_{P,c}$ to name a few encountered earlier).

To simplify the above bound further (at the cost of introducing some looseness) we can plug in upper bounds on $N_{d-1}(\epsilon)$ into Theorem 4.1. We prove the following lemma in Appendix C, and it follows from a simple geometric argument. The following bound is not tight, but suffices for subsequent asymptotic analysis. We use the bound presented in Corollary 4.2 overe the bound $N_{d-1}(\epsilon) \leq \left(\frac{3}{\epsilon}\right)^d$ (which follows from Lemma 5.7 of Wainwright [57]) due to slightly improved dependence on d in the exponent. If desired, one could obtain an analogue of Corollary 4.3 using the aforementioned bound as well, or even any bound on $N_{d-1}(\epsilon)$.

---

[2]Recall $N_{d-1}(\epsilon)$ was defined to be the ϵ-covering number of $\mathbb{S}^{d-1}$.

Lemma 4.2. *Let $\epsilon \in (0,1)$ be arbitrary and $d \geq 1$. Then,*

$$N_{d-1}(\epsilon) \leq C_d \left(\frac{3}{\epsilon} \right)^{d-1},$$

where C_d is a constant that does not depend on ϵ.

With the above bound on the covering number of $\mathbb{S}^{d-1}$, we have the following corollary.

Corollary 4.3. *Assume the same setup as in Theorem 4.1. Define $L_\rho^{cov} : \mathcal{L}_+(\mathbb{R}^d) \to \mathbb{R}_{\geq 0}$ by*

$$L_\rho^{cov}(V) := \log \left(h \left(\log_\alpha \left(\frac{\gamma_{\max}(V \vee \rho I_d)}{\rho} \right) \right) \right) + \log \left(\frac{C_d}{\delta(1 - \beta^{-1})} \right)$$
$$+ d \log \left(\frac{3\beta \sqrt{\kappa(V \vee \rho I_d)}}{\epsilon} \right).$$

Then,

$$\mathbb{P} \left(\exists t \geq 0 : \left\| (V_t \vee \rho I_d)^{-1/2} S_t \right\| \geq \frac{\sqrt{\gamma_{\min}(V_t \vee \rho I_d)}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\gamma_{\min}(V_t \vee \rho I_d)} L_\rho^{cov}(V_t) \right) \right) \leq \delta.$$

Proof. The result immediately follows by applying Theorem 4.1 and noting the bound

$$\beta \sqrt{\kappa(V \vee \rho I_d)} \cdot N_{d-1} \left(\frac{\epsilon}{\beta \sqrt{\kappa(V \vee \rho I_d)}} \right) \leq \frac{3\beta \sqrt{\kappa(V_t \vee \rho I_d)}}{\epsilon} \cdot \left(\frac{3\beta \sqrt{\kappa(V \vee \rho I_d)}}{\epsilon} \right)^{d-1}$$
$$= \left(\frac{3\beta \sqrt{\kappa(V \vee \rho I_d)}}{\epsilon} \right)^d,$$

which holds for all positive semi-definite matrices $V \in \mathcal{L}_+(\mathbb{R}^d)$. ∎

Treating the tuning parameters $\alpha, \beta, \epsilon, \rho$ as constants and selecting $h : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ satisfying $h(k) = O(\log(k))$ (which, as noted by Howard et al. [32], holds when $h(k) := (k+1)^s \zeta(s)$ for any $s > 1$), Corollary 4.3 yields that, with high probability, simultaneously for all $t \geq 0$,

$$\left\| V_t^{-1/2} S_t \right\| = O \left(\sqrt{\gamma_{\min}(V_t)} \cdot (\psi^*)^{-1} \left(\frac{1}{\gamma_{\min}(V_t)} \left[\log \log(\gamma_{\max}(V_t)) + d \log \kappa(V_t) \right] \right) \right).$$

We now specify (in terms of big-Oh notation) our bounds to the setting of two common CGF-like functions. First, we consider the case $\psi(\lambda) = \psi_{G,c}(\lambda) = \frac{\lambda^2}{2(1-c\lambda)}$, for which we recall that $\left(\psi_{G,c}^* \right)^{-1}(x) = \sqrt{2x} + cx$, and so our bounds yield that

$$\left\| V_t^{-1/2} S_t \right\| = O \left(\sqrt{\log \log(\gamma_{\max}(V_t)) + d \log \kappa(V_t)} + \frac{c}{\sqrt{\gamma_{\min}(V_t)}} \left[\log \log(\gamma_{\max}(V_t)) + d \log \kappa(V_t) \right] \right)$$

Further specifying to the case $\psi(\lambda) = \psi_N(\lambda) = \frac{\lambda^2}{2}$ (which is equivalent to the case $\psi = \psi_{G,c}$ with $c = 0$), the above bound reduces to the form:

$$\left\| V_t^{-1/2} S_t \right\| = O \left(\sqrt{\log \log(\gamma_{\max}(V_t)) + d \log \kappa(V_t)} \right). \tag{2}$$

The bound (2), in particular, captures the asymptotic growth rate of very general classes of sub-ψ process when $\psi(\lambda) \sim \frac{\lambda^2}{2}$ as $\lambda \downarrow 0$ (in a sense that we will make fully precise soon).

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4.1 Comparison With Existing Bounds

Now that we have presented the main result of the paper in Theorem 4.1 and examined the dimensional dependence of the bound in Corollary 4.3, we can compare our bounds to existing results in the literature. As a warm up (and sanity check), we ensure our bound presented in Theorem 4.1 recovers the results presented in Theorem 3.1, up to multiplicative or additive constants.

Comparison with Scalar Bounds: If $(S_t, V_t)_{t \geq 0}$ is sub-ψ and taking values in $\mathbb{R}$, we note that for any $\epsilon \in (0,1)$, $N_0(\epsilon) = 2$. Thus, Theorem 4.1 yields that, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$ (assuming $V_t \geq \rho$ for simplicity),

$$
\begin{aligned}
\frac{S_t}{\sqrt{V_t}} &\leq \sqrt{V_t} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{V_t} L_\rho(V_t) \right) \\
&= \frac{1}{1-\epsilon} \sqrt{V_t} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{V_t} \left[\ell_\rho(V_t) + \log \left(\frac{1}{1 - \beta^{-1}} \right) + \log (2\beta) \right] \right) \\
&= C_1 \sqrt{V_t} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{V_t} [\ell_\rho(V_t) + C_2] \right),
\end{aligned}
$$

where we have defined $C_1 := \frac{1}{1-\epsilon}$ and $C_2 := \log \left(\frac{1}{1-\beta^{-1}} \right) + \log (2\beta)$ for convenience. On the other hand, Theorem 3.1 yields that, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$ such that $V_t \geq \rho$,

$$
\frac{S_t}{\sqrt{V_t}} \leq \sqrt{V_t} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{V_t} \ell_\rho(V_t) \right).
$$

As expected, the bound yielded by Theorem 4.1 is looser than the that of Theorem 3.1, due to the extra union bound needed in the covering argument. However, the looseness is only by an absolute multiplicative factor C_1 and an additive factor of C_2 inside $(\psi^*)^{-1}$. Since the $N_0(\epsilon) = 2$ for all $\epsilon \in (0,1)$, the multiplicative factor can be forced to be arbitrarily small by appropriately choosing the covering parameter ϵ (e.g. sending $\epsilon \downarrow 0$ yields $C_1 \downarrow 1$). Likewise, the impact of the additive factor becomes vanishingly small as $V_t \to \infty$.

Method of Mixtures Bounds: Next, we compare our multivariate, self-normalized bounds to the "method of mixtures" bounds for sub-Gaussian concentration, in particular the following bound that follows from Example 4.2 of de la Peña et al. [21] and Theorem 1 of Abbasi-Yadkori et al. [2] and has become a staple in constructing confidence sets in online learning tasks [37, 61, 25]. We rephrase their sub-Gaussian result in the setting of "sub-ψ_N" concentration to ease comparison with our results.

Fact 4.4 (de la Peña et al. [21], Abbasi-Yadkori et al. [2])**.** *Let $(S_t, V_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued sub-ψ_N process where $V_t = \sum_{s=1}^{t} \mathbb{E}_{s-1} \Delta S_s \Delta S_s^\top$. Then, for any $\delta \in (0,1)$ and any $\rho > 0$, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$,*

$$
\left\| (V_t + \rho I_d)^{-1/2} S_t \right\| \leq \sqrt{2 \log \left(\frac{1}{\delta} \sqrt{\det (I_d + \rho^{-1} V_t)} \right)}.
$$

We note that the above bound holds *only* in the case where the process $(S_t)_{t \geq 0}$ has sub-Gaussian increments, and it is not obvious whether or not a similar result holds for other tails, for more CGF-like functions ψ, and adapted (not predictable) V_t. In the case $\psi = \psi_N$, as noted in (2), our bound is of the form $\left\| V_t^{-1/2} S_t \right\| = O\left(\sqrt{\log\log(\gamma_{\max}(V_t)) + d \log \kappa(V_t)} \right)$. These two bounds (those based on the determinant of the variance proxy and those based on the condition number of the variance proxy) are fundamentally incomparable. When V_t is well-conditioned, we expect our bounds to be tighter than the bound in Fact 4.4, as our bounds will be of order $\approx \sqrt{\log\log(\gamma_{\max}(V_t)) + d}$. If $\kappa(V_t) \approx \gamma_{\max}(V_t)$, we may expect the determinant rate bound in Fact 4.4 to be tighter, as the bound provided by Theorem 4.1 will be of order $\approx \sqrt{\log\log(\gamma_{\max}(V_t)) + d \log(\gamma_{\max}(V_t))}$, and $d \log \gamma_{\max}(V_t) \geq \log\det(V_t)$ (ignoring the shift ρ in the covariance matrix). One particularly useful feature of our bounds is that the do not require a shift in variance proxy as the bound in Fact 4.4 does. It is an interesting open problem to derive determinant-rate self-normalized bounds under more general tail conditions and for adapted (not predictable) V_t.

Backwards Martingale Bounds: As a last point of comparison, we relate our bounds to the recent bounds constructed by Manole and Ramdas [44] using backwards or reverse martingale techniques. We note that the bounds of Manole and Ramdas [44] hold for *any* fixed norm on $\mathbb{R}^d$ (e.g. ℓ_p norms, for instance), but we only present the result in the case of the ℓ_2 norm, as this is the setting in which our bounds are comparable. The authors leverage the following bounds in estimating an unknown, multivariate mean from i.i.d. data. In our statement below, we center all observations so that the unknown mean always takes value zero for ease of comparison.

Fact 4.5 (Corollary 23 of Manole and Ramdas [44])**.** *Let* $S_t := \sum_{s=1}^{t} X_s$, *where* $(X_t)_{t \geq 0}$ *are i.i.d. with mean 0. Let* $h : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ *satisfy* $\sum_{k=0}^{\infty} h(k)^{-1} \leq 1$, *and let* $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ *be CGF-like. Suppose that, for any* $\lambda \in [0, \lambda_{\max})$ *and* $t \geq 0$, $\sup_{\nu \in \mathbb{S}^{d-1}} \log \mathbb{E} e^{\lambda \langle \nu, X_t \rangle} \leq \psi(\lambda)$. *Then, for any* $\delta \in (0, 1)$, *with probability at least* $1 - \delta$, *simultaneously for all* $t \geq 0$,

$$\| S_t / \sqrt{t} \| \leq \frac{\sqrt{t}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{2}{t} \left[\log(h(\log_2(t))) + \log\left(\frac{1}{\delta}\right) + \log N_{d-1}(\epsilon) \right] \right).$$

It is clear that the process $(S_t)_{t \geq 0}$ is sub-ψ with variance proxy $(V_t)_{t \geq 0}$ given by $V_t := t I_d$, and so Theorem 4.1 (taking $\rho = 1$) applied to this setting yields that, with probability at least $1 - \delta$, simultaneously for all $t \geq 1$,

$$\| S_t / \sqrt{t} \| \leq \frac{\sqrt{t}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{t} \left[\log(h(\log_\alpha(t))) + \log\left(\frac{\beta}{\delta(1 - \beta^{-1})}\right) + \log N_{d-1}(\epsilon) \right] \right).$$

In this particular setting, our bound is almost equivalent to that of Manole and Ramdas [44], being looser is a vanishingly small additive factor $\log\left(\frac{\beta}{1 - \beta^{-1}}\right)$ due to the covering argument needed to control the geometric "distortions" induced by the variance proxy $(V_t)_{t \geq 0}$. However, we note that our bound is significantly more general, as it allows for arbitrary martingale dependence between observed random variables. This is in contrast to the bound of Manole and Ramdas [44], as this bound is only valid if the data are known to be i.i.d. (or,

at the very least, exchangeable). The argument used by Manole and Ramdas [44] does not readily generalize to general dependence structures because they leverage reverse martingales in the exchangeable filtration, thus requiring that the data be exchangeable.

4.2   Vector Laws of the Iterated Logarithm

In Corollary 3.2, we discussed how our scalar bounds can be used to derive a version of the law of the iterated logarithm for scalar sub-ψ processes. In particular, this bound obtained the optimal constant matching the case of i.i.d. random variables (see Durrett [26], Chapter 8 or Howard et al. [32]), showing that our bounds are unimprovable asymptotically.

In the multivariate setting, our bounds do not just depend on $\log \log(\gamma_{\max}(V_t))$, but also on $\log \kappa(V_t)$. This dependence is not simply an artefact of our analysis, as de la Peña et al. [21] show an example of a 2-dimensional process $(S_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ satisfying $\|V_t^{-1/2} S_t\| \sim \sqrt{\log \kappa(V_t)}$ almost surely.

In this section, we aim to show that our results are asymptotically optimal in the following sense. First, we show that, under a simple set of assumptions, if $(S_t, V_t)_{t \geq 0}$ is a sub-ψ, then $\limsup_{t \to \infty} \frac{\|V_t^{-1/2} S_t\|}{\sqrt{2 \log \log(\gamma_{\max}(V_t)) + d \log \kappa(V_t)}} \leq 1$ almost surely. Secondly, we show that this bound is "tight" in the sense that there exists a sub-ψ process $(S_t, V_t)_{t \geq 0}$ such that $\|V_t^{-1/2} S_t\| = \Theta(\sqrt{\log \log \gamma_{\max}(V_t) + d \log \kappa(V_t)})$ almost surely.

We start by presenting the first result, which can be viewed as an "upper law of the iterated logarithm". We prove this result in Section 7.

Corollary 4.6. *Let $(S_t)_{t \geq 0}$ be an $\mathbb{R}^d$-valued sub-ψ process with variance proxy $(V_t)_{t \geq 0}$. Suppose that (a) $\psi(\lambda) \sim \frac{\lambda^2}{2}$ as $\lambda \downarrow 0$, (b) $\gamma_{\min}(V_t) \xrightarrow[t \to \infty]{} \infty$ almost surely, and (c) and $\log(\gamma_{\max}(V_t))/\gamma_{\min}(V_t) = o(1)$ almost surely. Then,*

$$\limsup_{t \to \infty} \frac{\|V_t^{-1/2} S_t\|}{\sqrt{2 \log \log \gamma_{\max}(V_t) + d \log \kappa(V_t)}} \leq 1$$

almost surely.

We can compare the above corollary to the discussion at the beginning of Section 3 of de la Peña et al. [21], where the authors show that when $(S_t)_{t \geq 0}$ and $(V_t)_{t \geq 0}$ satisfy certain assumptions based on finiteness of pth moments, one has

$$\limsup_{t \to \infty} \frac{\|(V_t + V)^{-1/2} S_t\|}{\sqrt{\log \log \gamma_{\max}(V + V_t) + \log \kappa(V_t + V)}} = O(1) \qquad \text{almost surely,}$$

where the constant masked by the "Big-Oh" notation maybe be random. Our bound is more precise than their bound in that (a) we obtain an explicit constant in our asymptotic bound, (b) the bound recovers the LIL in the case $d = 1$ (see the earlier discussed Corollary 3.2), and (c) our bound elicits explicit dependence on the ambient dimension d.

The remaining question is if the above law of the iterated logarithm is tight. As afore-mentioned, de la Peña et al. [21] show the existence of a two-dimensional process satisfying $\|V_t^{-1/2} S_t\| \sim \log \kappa(V_t)$ almost surely. We first describe this example, and then show how

to extend it to higher dimensions. In particular, we will construct a process that attains the same rate as the upper bound presented in our Corollary 4.6, up to a small, absolute constant. We start by describing the example of de la Peña et al. [21].

Example 4.7. Let $(\epsilon_t)_{t\geq 1}$ be a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables, and let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration associated with $(\epsilon_t)_{t\geq 1}$. First, define the regressors $(U_t)_{t\geq 1}$ by $U_1 = 0$ and $U_{t+1} := \overline{U}_t + \bar{\epsilon}_t$, where for a sequence $(y_t)_{t\geq 1}$ we define $\bar{y}_t := \frac{1}{t}(y_1 + y_2 + \cdots + y_t)$. Then, embed these regressors into $\mathbb{R}^2$ by defining the process $(X_t)_{t\geq 1}$ as $X_t := (1, U_t)^\top$. Clearly, by construction, the process $(X_t)_{t\geq 1}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable.

With these sequentially constructed regressors, one can construct a martingale $(S_t)_{t\geq 0}$ with respect to $(\mathcal{F}_t)_{t\geq 0}$ given by $S_t := \sum_{s=1}^t \epsilon_s X_s$ and a corresponding predictable covariance process $(V_t)_{t\geq 0}$ given by $V_t = \sum_{s=1}^t X_s X_s^\top$. de la Peña et al. [21] show that the following hold almost surely:

1. $\gamma_{\max}(V_t) \sim t\,(1 + \sum_{s=1}^\infty s^{-1}\epsilon_s)$,

2. $\gamma_{\min}(V_t) \sim \frac{log(t)}{1+\sum_{s=1}^\infty s^{-1}\epsilon_s}$, and

3. $\|V_t^{-1/2}S_t\| \sim \sqrt{\log(t)}$.

Noting that $\log \kappa(V_t) = \log(\gamma_{\max}(V_t)/\gamma_{\min}(V_t)) \sim \log(t)$, we see that we have $\|V_t^{-1/2}S_t\| \sim \sqrt{\log \kappa(V_t)}$ almost surely. Further, it is easily checked that $(S_t, V_t)_{t\geq 0}$ is sub-ψ_N, per Definition 2.2. Thus, this example shows that the logarithmic dependence on $\kappa(V_t)$ in Theorem 4.1 cannot, in general, be dropped.

While the above example demonstrates the inevitability of having $\log \kappa(V_t)$ appear in non-asymptotic, self-normalized concentration for vector-valued processes, it does not capture dependence on dimensionality. In the next example, we show that there exists sub-ψ processes $(S_t, V_t)_{t\geq 0}$ such that $\|V_t^{-1/2}S_t\| \sim \sqrt{\frac{d}{2} \log \kappa(V_t)}$, showing our upper bounds are within a multiplicative factor $\sqrt{2}$ of optimal.

Example 4.8. Suppose d is even. Let $(S_t^{(1)})_{t\geq 0}, \ldots, (S_t^{(d/2)})_{t\geq 0}$ be i.i.d. copies of the process constructed in Example 4.7, $(V_t^{(1)})_{t\geq 0}, \ldots, (V_t^{(d/2)})_{t\geq 0}$ the corresponding predictable covariance processes, and $(\mathcal{F}_t)_{t\geq 0}$ the smallest filtration for which $(\epsilon_t^{(1)})_{t\geq 1}, \ldots, (\epsilon_t^{(d/2)})_{t\geq 1}$ are adapted, i.e. the filtration given by $\mathcal{F}_t := \mathcal{F}_t^{(1)} \bigvee \cdots \bigvee \mathcal{F}_t^{(d/2)}$, where $\mathcal{F} \bigvee \mathcal{G}$ denotes the "join" of σ-algebras $\mathcal{F}, \mathcal{G}$, i.e. the smallest σ-algebra containing both.

Define the $\mathbb{R}^d$-valued process $(S_t)_{t\geq 0}$ by $S_t := \left(S_t^{(1)}, \ldots, S_t^{(d/2)} \right)$, and the corresponding covariance process $(V_t)_{t\geq 0}$ by

$$
V_t := \begin{pmatrix} V_t^{(1)} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & V_t^{(2)} & \cdots & \mathbf{0} \\ \vdots & & \ddots & \mathbf{0} \\ \mathbf{0} & \cdots & \cdots & V_t^{(d/2)} \end{pmatrix}.
$$

Clearly $(S_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted and $(V_t)_{t\geq 0}$ is $(\mathcal{F}_t)_{t\geq 0}$-predictable. Moreover, it can readily be checked that $(S_t, V_t)_{t\geq 0}$ is a sub-ψ_N process.

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Since V_t is a block-diagonal matrix, we clearly have $\gamma_{\max}(V_t) = \max_{i \in [d/2]} \gamma_{\max}(V_t^{(i)})$ and $\gamma_{\min}(V_t) = \min_{i \in [d/2]} \gamma_{\min}(V_t^{(i)})$. Thus, using the reasoning on the almost sure behavior on $\gamma_{\max}(V_t^{(i)})$ and $\gamma_{\min}(V_t^{(i)})$ presented in Example 4.7, we see that $\log \kappa(V_t) \sim \log(t)$ almost surely. Further, it isn't hard to see that

$$\|V_t^{-1/2} S_t\|^2 = S_t^\top V_t^{-1} S_t$$
$$= (S_t^{(1)})^\top (V_t^{(1)})^{-1} S_t^{(1)} + \cdots + (S_t^{(d/2)})^\top (V_t^{(d/2)})^{-1} S_t^{(d/2)} \sim \frac{d}{2} \log \kappa(V_t).$$

Thus, we have shown that, up to small constants, the dependence on $\log \kappa(V_t)$ and d in Theorem 4.1 (and thus the corresponding dependence in Corollary 4.6) is unimprovable.

5 Applications to Online Linear Regression

We now use our self-normalized bounds to construct confidence ellipsoids for slope estimation in online linear regression. In online linear regression, a statistician interacts with an environment over a sequence of rounds. At the beginning of each round, he adaptively (perhaps using observations from previous rounds) selects a point $X_t \in \mathbb{R}^d$, and then observes noisy feedback $Y_t := \langle X_t, \theta^* \rangle + \epsilon_t$, where ϵ_t represents some mean zero noise variable and θ^* is a fixed slope vector. The goal of the statistician is to produce a *confidence sequence* for the unknown slope vector — that is, a time indexed sequences of sets that all simultaneously contain the unknown parameter with high probability. We formalize the online linear regression model as follows.

Model 5.1 (Online Linear Regression). *Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration and $\theta^* \in \mathbb{R}^d$ a fixed (unknown) slope vector. The online linear regression model is characterized by three processes: (a) a $(\mathcal{F}_t)_{t \geq 0}$-predictable $\mathbb{R}^d$-valued sequence $(X_t)_{t \geq 1}$ representing adaptively-chosen covariates, (b) a $(\bar{\mathcal{F}}_t)_{t \geq 0}$-adapted scalar-valued processes $(\epsilon_t)_{t \geq 1}$ representing noise, and (c) $(Y_t)_{t \geq 1}$ given as $Y_t = \langle X_t, \theta^* \rangle + \epsilon_t$ representing noisy responses. We assume the residual process $S_t := \sum_{s=1}^t \epsilon_s X_s$ is sub-ψ with (predictable) variance proxy $V_t := \sum_{s=1}^t X_s X_s^\top$, where ψ is a super-Gaussian CGF-like function.*

For a fixed regularization parameter $\rho > 0$, we consider the sequence of **least squares estimates with shrinkage** given by

$$\widehat{\theta}_t := (\mathbf{X}_t^\top \mathbf{X}_t \vee \rho I_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t, \quad t \geq 1,$$

where $\mathbf{X}_t \in \mathbb{R}^{t \times d}$ has $X_1, \ldots, X_t$ as its rows and $\mathbf{Y}_t \in \mathbb{R}^d$ is a column vectors with $Y_1, \ldots, Y_t$ as its entries. Clearly $\widehat{\theta}_t$ reduces to the standard least-squares estimator when $\gamma_{\min}(\mathbf{X}_t^\top \mathbf{X}_t) \geq \rho$.

The assumption that (S_t, V_t) is sub-ψ is often mild. For example, it is satisfied (a) if $\log \mathbb{E}_{t-1} \exp\{\lambda \epsilon_t\} \leq \psi_N(\lambda)$ for all $\lambda \in \mathbb{R}$ (i.e. ϵ_t is conditionally sub-Gaussian), or (b) if $\|X_t\| \leq 1$ for all $t \geq 1$ and $\log \mathbb{E}_{t-1} \exp\{\pm \lambda \epsilon_t\} \leq \psi(\lambda)$ for some super-Gaussian ψ and all $\lambda \in [0, \lambda_{\max})$. We prove this in Proposition B.1 in Appendix B. The assumption that $\|X_t\| \leq 1$ for all $t \geq 1$ in the above can be replaced with the assumption that $\|X_t\| \leq R$ for any fixed $R > 0$ by appropriate rescaling. This type of boundedness assumption is regularly made in the mult-armed bandit literature [2, 12, 42], and thus has practical relevance.

5.1 Time-Uniform Confidence Ellipsoids

We briefly discuss how confidence ellipsoids are constructed in classical least-squares regression. In this setting, one observes a matrix of covariates $\mathbf{X} \in \mathbb{R}^{t \times d}$ and a response vector $\mathbf{Y} \in \mathbb{R}^t$ given by $\mathbf{Y} = \mathbf{X}\theta^* + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \sigma^2 I_d)$. If $\mathbf{X}^\top \mathbf{X}$ is full rank, it is well-known [49, 36] that the least-squares estimate for θ^*, given by $\widehat{\theta} := (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{Y}$, satisfies

$$\sigma^{-1} \|(\mathbf{X}^\top \mathbf{X})^{1/2}(\widehat{\theta} - \theta^*)\| \sim \chi_q^2,$$

where χ_q^2 denotes the Chi-squared distribution with q degrees of freedom. Letting $x_{q,\delta}$ denote its δth upper quantile[3] it follows that the set

$$\mathcal{C} := \{\theta \in \mathbb{R}^d : \sigma^{-1} \|(\mathbf{X}^\top \mathbf{X})^{1/2}(\widehat{\theta} - \theta)\| \leq x_{q,\delta}\}$$

forms an exact $1 - \delta$ confidence ellipsoid for θ^* centered at $\widehat{\theta}$.

The above confidence ellipsoid fails to be valid when $\mathbf{X}$ is no longer fixed or when the added noise variables are no longer i.i.d. Gaussian, which is the case presented in our heuristic model above. To circumvent this failure of classical statistical machinery, we can leverage our self-normalized bounds for vector-valued processes to construct confidence ellipsoids for θ^* that are valid across all time steps uniformly. We do exactly this in the following theorem.

Theorem 5.2. *Consider Model 5.1, let $\delta \in (0, 1)$ be arbitrary and set $V_t := \mathbf{X}_t^\top \mathbf{X}_t = \sum_{s=1}^t X_s X_s^\top$. Then, with probability at least $1 - \delta$, simultaneously for all $t \geq 1$, we have*

$$\|(V_t \vee \rho I_d)^{1/2}(\widehat{\theta}_t - \theta^*)\| < \frac{\sqrt{\gamma_{\min}(V_t \vee \rho I_d)}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\gamma_{\min}(V_t \vee \rho I_d)} L_\rho(V_t) \right) + \sqrt{\rho} \|\theta^*\| \mathbb{1}_{\gamma_{\min}(V_t) < \rho},$$

where the parameters $\alpha, \epsilon, \beta, h$ and the function L_ρ (which partially masks parameter dependence) are as outlined in Theorem 4.1.

The dependence on the norm of the unknown slope vector in Theorem 5.2 may be a mild irritant, but note that the indicator function multiplying it ensures that the term necessarily disappears for large t. Also note that assuming a known bound on $\|\theta^*\|$ is common in many statistical tasks, in particular those related to bandit optimization [2, 12, 42]. Nonetheless, this dependence can be fully removed by sacrificing validity of the bound above bound when the minimum eigenvalue of $\mathbf{X}_t^\top \mathbf{X}_t$ is small. This is made precise as follows.

Corollary 5.3. *Assume the same setup as Theorem 5.2. Then, we have*

$$\mathbb{P} \left(\exists t \geq 0 : \|V_t^{1/2}(\widehat{\theta}_t - \theta^*)\| \geq \frac{\sqrt{\gamma_{\min}(V_t)}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\gamma_{\min}(V_t)} L_\rho(V_t) \right) \text{ and } V_t \succeq \rho I_d \right) \leq \delta.$$

We can similarly derive a result for the ridge estimators of the unknown slope parameter, which are given by $\widetilde{\theta}_t := (\mathbf{X}_t^\top \mathbf{X}_t + \rho I_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t$. We prove the following in Appendix B.

---

[3]that is, $x_{q,\delta} > 0$ is the unique value satisfying $\mathbb{P}(X \geq x_{q,\delta}) = \delta$, where $X \sim \chi_q^2$

Corollary 5.4. *Assume the setup outlined in Theorem 5.2 above. Consider the sequence of ridge estimates $(\widetilde{\theta}_t)_{t \geq 1}$ given by*

$$\widetilde{\theta}_t := (\mathbf{X}_t^\top \mathbf{X}_t + \rho I_d)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t.$$

Set $V_t := \mathbf{X}_t^\top \mathbf{X}_t$. With probability at least $1 - \delta$, simultaneously for all $t \geq 1$, we have

$$\|(V_t + \rho I_d)^{1/2}(\widetilde{\theta}_t - \theta^*)\| < \frac{\sqrt{\gamma_{\min}(V_t + \rho I_d)}}{1 - \epsilon} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\gamma_{\min}(V_t + \rho I_d)} L_\rho(V_t + \rho I_d)\right) + \sqrt{\rho}\|\theta^*\|,$$

where the parameters $\alpha, \epsilon, \beta, h$ and the function L_ρ (which partially masks parameter dependence) are as outlined in Theorem 4.1.

Comparison with Existing Bounds: Many results concerning finite-sample properties of regression estimators are based either in the setting of fixed design [57, 4] or in the the case of independent covariates [38, 39]. Moreover, these results are more often than not concerned with bounding the ℓ_2-error of the estimator, i.e. the quantity $\|\widehat{\theta}_t - \theta^*\|$, as opposed to the self-normalized quantities we study.

The main points of comparison for our results have been derived in the online learning/regression literature. We compare our results to those of Abbasi-Yadkori et al. [2]. In their work, Abbasi-Yadkori et al. [2] construct a confidence sequence for estimating an unknown slope vector θ^* by utilizing self-normalized concentration for sub-Gaussian processes (in particular, leveraging a Gaussian mixture technique that dates back to Example 4.2 in de la Peña et al. [21]). While subsequent confidence sequences have been derived in the setting of regression with variance estimation [25], semiparametric regression with bounded confounding [37], and ridge regression in reproducing kernel Hilbert spaces [61, 1], we focus just on the original contributions of Abbasi-Yadkori et al. [2] since all subsequent results exhibit the same rate and hold only in the setting of sub-Gaussian noise.

Fact 5.5 (Theorem 2 of Abbasi-Yadkori et al. [2]). *Let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration, let $(X_t)_{t \geq 1}$ be an $(\mathcal{F}_t)_{t \geq 0}$-predictable sequence in $\mathbb{R}^d$, and let $(\epsilon_t)_{t \geq 1}$ be a real-valued $(\mathcal{F}_t)_{t \geq 1}$-adapted sequence such that conditional on $\mathcal{F}_{t-1}$, $\log \mathbb{E}_{t-1} \exp\{\lambda \epsilon_t\} \leq \psi_N(\lambda)$ for all $\lambda \in \mathbb{R}$. Then, for any $\rho > 0$ and $\delta \in (0, 1)$,*

$$\mathbb{P}\left(\exists t \geq 0 : \|(V_t + \rho I_d)^{1/2}(\widetilde{\theta}_t - \theta^*)\| \geq \sqrt{2\log\left(\frac{1}{\delta}\sqrt{\det(I_d + \rho^{-1}V_t)}\right)} + \sqrt{\rho}\|\theta^*\|\right) \leq \delta,$$

where $\widetilde{\theta}_t$ is the ridge regression estimator outlined in Corollary 5.4 and $V_t := \sum_{s=1}^{t} X_s X_s^\top$.

We compare our results to Fact 5.5 in the setting $\psi = \psi_N$, as this is the only setting in which the results of Abbasi-Yadkori et al. [2] are valid. We first qualitatively compare the above confidence sequence to the one presented in Corollary 5.4. Both bounds suffer the same dependence on the norm of the unknown slope vectors and differ only in the first term. Namely, as noted earlier, $(\psi_N^*)^{-1}(u) = \psi_N^{-1}(u) = \sqrt{2u}$, so in this setting our bound reduces to the form

$$\|(V_t + \rho I_d)^{1/2}(\widetilde{\theta}_t - \theta^*)\| \leq \sqrt{2\alpha L_\rho(V_t + \rho I_d)} + \sqrt{\rho}\|\theta^*\|$$

$$= O\left(\sqrt{\log\log\gamma_{\max}(\rho^{-1}V_t + I_d) + d\log\kappa(V_t + \rho I_d)}\right)$$

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simultaneously for all $t \geq 0$ with probability at least $1 - \delta$. Thus, the same comparison made in Subsection 4.1 applies in this setting — when the (shifted) covariance $V_t + \rho I_d$ is poorly conditioned, the bound presented in Fact 5.5 would be expected to be tighter. Likewise, when $V_t + \rho I_d$ is well-conditioned, Corollary 5.4 may be tighter.

A more interesting comparison is between Fact 5.5 and Theorem 5.2 (more specifically, Corollary 5.3 following the theorem statement). Whereas the above fact provides convergence guarantees for ridge regression estimates, Corollary 5.3 applies directly to the unregularized, least-squares estimates of the unknown slope vector. In particular, the bound does not depend on $\|\theta^*\|$, the norm of the unknown slope vector. This may be desirable in many statistical settings in which either advanced knowledge of such a bound is unavailable or only a loose bound on the quantity is known. Moreover, this bound is interesting in itself as no shift in covariance is required in constructing the confidence ellipsoids. The rate given by Corollary 5.3 is essentially the same that provided by Corollary 5.4 modulo the presence of a shift in the covariance matrix, i.e. the corollary yields that with high probability, uniformly in time, $\|V_t^{1/2}(\widehat{\theta}_t - \theta^*)\| = O\left(\sqrt{\log\log_{\max}(\rho^{-1}V_t) + d\log\kappa(V_t)}\right)$.

5.2 Applications to Vector Autoregressive Models

We now show how to apply our confidence ellipsoids from Subsection 5.1 in the section to a vector autoregressive model. We take inspiration from Bercu and Touati [8], who leverage self-normalized concentration results for scalar-valued processes to measure the convergence of least-squares and Yule-Walker estimates for a simple one stage autoregressive model (i.e. an AR(1) model). We focus solely on the least-squares estimates in the sequel. We provide a brief, high-level qualitative comparison between these results and our own. The following results may be of practical interest as autoregressive models and other time series models are frequently applied to problems in econometrics [4, 50] and finance [45, 19].

The results we provide in this section are more general than those of Bercu and Touati [8] in three ways. First, these authors assume that all noise variables are Gaussian, whereas we allow the noise to be instead conditionally sub-Gaussian. Second, we handle a vector autoregressive model, whereas Bercu and Touati [8] only handle the univariate case. Lastly, we handle the problem of general autoregression with p-stages of lag, whereas Bercu and Touati [8] only handle the case $p = 1$. Our bounds are also different than those of Bercu and Touati [8] in that they are derived in terms of the predictable covariance associated with observations, whereas those of Bercu and Touati [8] are stated in terms of total number of observations. With these comparisons in hands, we now describe the p-stage vector autoregressive model (hereinafter referred to as VAR(p) for short).

Model 5.6. *A* **p** *stage vector-valued autoregressive model, denoted by* VAR(p)*, is an $\mathbb{R}^d$-valued process $(Y_t)_{t \geq -p+1}$ such that $Y_{-p+1}, \ldots, Y_0 \in \mathbb{R}^d$ and $Y_t := \sum_{i=1}^p A_i Y_{t-i} + \epsilon_t$ where (a) $A_i \in \mathbb{R}^{d \times d}$ are fixed matrices for all $i \in [p]$, and (b) ϵ_t satisfies $\log \mathbb{E}_{t-1} \exp\{\lambda\langle\nu, \epsilon_t\rangle\} \leq \frac{\lambda^2}{2}$, where $\nu \in \mathbb{S}^{d-1}$ and $\lambda \in [0, \lambda_{\max})$. In the above, $\mathbb{E}_{t-1}[\cdot] := \mathbb{E}(\cdot \mid \mathcal{F}_{t-1})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the filtration given by $\mathcal{F}_t := \sigma(Y_s : -p + 1 \leq s \leq t)$, for any $t \geq 1$.*

For more details on vector autoregressive models, see [29]. In words, a process $(Y_t)_{t \geq 0}$ satisfies the conditions of a p-stage autoregressive (or VAR(p)) model if Y_t is a linear function

of $Y_{t-1}, \ldots, Y_{t-p}$ plus mean zero noise. In the above, the values $Y_{-p+1}, \ldots, Y_0$ are treated as fixed nonrandom vectors, as is typical in much of the time series analysis literature. However, all results in the sequel still hold if $Y_{-p+1}, \ldots, Y_0$ are random variables that are independent of the noise sequence $(\epsilon_t)_{t\geq 1}$. Typically, the VAR(p) model also admits a vector mean parameter $\mu \in \mathbb{R}^d$, having the relationship $Y_t = \mu + \sum_{i=1}^{p} A_p Y_{t-p} + \epsilon_t$ for all $t \geq 1$, but we omit this to simplify exposition.

The goal of the statistician running an autoregressive model is twofold: (a) to estimate the unknown matrix parameters $A_1, \ldots, A_p$, and (b) to calibrate confidence in his estimates. Before discussing classical approaches to estimating these parameters, we simplify notation. We define the "stacked" transition matrix $\Pi \in \mathbb{R}^{d \times dp}$ and process vectors $(X_t)_{t\geq 1} \in \mathbb{R}^{dp}$ by

$$\Pi := (A_1, \ldots, A_p) \quad \text{and} \quad X_t := (Y_{t-1}, Y_{t-2}, \ldots, Y_{t-p})^\top.$$

For $i \in [d]$, we denote by $\pi(i) \in \mathbb{R}^{dp}$ the ith row of the stacked matrix Π. We likewise denote by $\epsilon_t(i) \in \mathbb{R}$ the ith component of the noise vector ϵ_t and $X_t(i)$ the ith component of the state vector X_t. Let $\mathbf{X}_t \in \mathbb{R}^{t \times dp}$ be the matrix with $X_1, \ldots, X_t$ as its rows, and let $\mathbf{Y}_t \in \mathbb{R}^{t \times d}$ have $Y_1, \ldots, Y_t$ as its rows. For $i \in [d]$, let $\mathbf{Y}_t(i) \in \mathbb{R}^t$ denote the ith column of $\mathbf{Y}_t$.

If $(\epsilon_t)_{t\geq 1}$ are i.i.d. $\mathcal{N}(0, \sigma^2 I_d)$ with known standard deviation σ, it is well-known (see Hamilton [29], Chapter 11) that the maximum likelihood estimate for Π at time $t \geq 1$, for now denoted $\widehat{\Pi}_t$, has rows $\widehat{\pi}_t(i)$ that are just the least-squares estimates given by

$$\widehat{\pi}_t(i) := \left(\mathbf{X}_t^\top \mathbf{X}_t\right)^{-1} \mathbf{X}_t^\top \mathbf{Y}_t(i). \tag{3}$$

It thus makes sense to study the convergence on these row-wise estimates in the remainder of this section. We focus on studying the convergence of a single row estimate, as the general case follows from union-bounding over the validity of the d row estimates. The proof of the following is a straightforward consequence of Theorem 5.2, and we provide the brief proof of the result in Appendix B.

Corollary 5.7. *For a fixed coordinate $i \in [d]$, let $(\widehat{\pi}_t(i))_{t\geq 1}$ be the sequence of estimates outlined in (3). Let $\rho > 0$ and $\delta \in (0,1)$ be arbitrary. Define the covariance process $(V_t)_{t\geq 1}$ by $V_t := \mathbf{X}_t^\top \mathbf{X}_t$. Then, with probability at least $1 - \delta$, simultaneously for all $t \geq 1$, we have*

$$\|(V_t \vee \rho I_{dp})^{1/2}(\widehat{\pi}_t(i) - \pi(i))\| \leq \frac{1}{1-\epsilon}\sqrt{2\alpha L_\rho(V_t)} + \sqrt{\rho}\|\theta^*\|\mathbb{1}_{\gamma_{\min}(V_t)<\rho},$$

where the parameters $\alpha, \epsilon, \beta, h$ and the function L_ρ (which partially masks parameter dependence) are as outlined in Theorem 4.1.

We now compare Corollary 5.7 to traditional asymptotic analyses of equation estimation in the VAR(p) model. First, note that, in Model 5.6 and Corollary 5.7, we place no assumptions on the matrices $A_1, \ldots, A_p \in \mathbb{R}^{d \times d}$. This is in contrast to typical asymptotic analyses, which must assume that all solutions $z \in \mathbb{C}$ to the equation

$$\det(I_d + A_1 z + A_2 z^2 + \cdots + A_p z^p) = 0 \tag{4}$$

have modulus $|z| > 1$ (which we assume holds for validity of the following comparison). In the setting of independent Gaussian noise, as discussed above, the stacked process $(X_t)_{t\geq 1}$ is ergodic and admits some stationary distribution π over $\mathbb{R}^{dp}$. It is known is known that, for any

$i \in [d]$, $\sqrt{t} V^{1/2}(\widehat{\pi}_t(i) - \pi(i)) \Rightarrow \mathcal{N}(0, \sigma^2 I_{dp})$, where $V = \mathbb{E}_\pi[X_t X_t^\top] = \lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^t X_s X_s^\top$ (the final equality comes from the ergodicity of $(X_t)_{t \geq 1}$). For large t, one would thus expect that $\|V_t^{1/2}(\widehat{\pi}_t(i) - \pi(i))\| \lesssim \sqrt{dp}$ with high probability.

We compare our non-asymptotic bounds to this rate. Observe that, Corollary 5.7 yields that, with high-probability, simultaneously for all $t \geq 1$,

$$\|V_t^{1/2}(\widehat{\pi}_i(t) - \pi(i))\| = O\left(\sqrt{dp \log \kappa(V_t) + \log \log (\gamma_{\max}(V_t))}\right).$$

If $\lim_{t \to \infty} \frac{1}{t} \sum_{s=1}^t X_s X_s^\top = V$ for some fixed positive-definite matrix V (as will be the case if the ϵ_t are i.i.d.) and t is a sufficiently large "target round", we can view the above as stating $\|V_t^{1/2}(\widehat{\pi}_i(t) - \pi(i))\|$ is bounded above by a term growing like $O(\sqrt{\log \log \gamma_{\max}(t) + dp})$ (since $\kappa(V_t) = \kappa(V) = O(1)$ for large t, almost surely). As expected in time-uniform concentration, the bounds presented in Corollary 5.7 are looser than those provided by the central limit theorem by a doubly logarithmic factor.

Comparison with Existing Bounds: We lastly make a brief comparison with the bounds of Bercu and Touati [8] in the univariate case. In this case, the autoregressive model is parameterized by a scalar $a \in \mathbb{R}$ instead of a sequence of matrices. We thus denote the least-squares estimator of a at time $t \geq 1$ as $\widehat{a}_t := \frac{\sum_{s=1}^t X_{t-1} X_t}{\sum_{s=1}^t X_{t-1}^2}$, departing from our notation of $\widehat{\pi}_t$, which was relevant for estimating a row in a stacked matrix. We state the bound of Bercu and Touati [8] for convenience.

Fact 5.8 (Corollary 5.2 of Bercu and Touati [8]). *Suppose $a \in \mathbb{R}$ is fixed. Further, suppose $(Y_t)_{t \geq 0}$ is given by $Y_0 \sim \mathcal{N}(0,1)$ and $Y_t := aY_{t-1} + \epsilon_t$, where $(\epsilon_t)_{t \geq 1}$ are a sequence of i.i.d. $\mathcal{N}(0,1)$ random variables independent of Y_0. Then, for any fixed $x > 0$ and $t \geq 1$, we have*

$$\mathbb{P}\left(|\widehat{a}_t - a| \geq x\right) \leq 2 \exp\left\{\frac{-tx^2}{2(1 + y_x)}\right\},$$

where y_x is the unique solution to the equation $\psi_{P,1}^*(y_x) = x^2$, where we recall $\psi_{P,1}^*(u) = (1+u)\log(1+u) - u$.

We draw several high-level comparisons between the bounds. First, the bound in Corollary 5.7 is self-normalized, being defined in terms of the empirical variance $V_t = \sum_{s=1}^t Y_{s-1}^2$. The bound in Fact 5.8, on the other hand, depends just on the number of samples used to construct the least-squares estimator, and thus is not self-normalized. Another difference between the conclusions of Fact 5.8 and Corollary 5.7 is that Fact 5.8 holds only for an individual, fixed sample size $t \geq 1$ whereas Corollary 5.7 is valid for all $t \geq 1$ *simultaneously*. To use Fact 5.8 to obtain a time-uniform guarantee, one would need to use a union bound argument to allocate the total failure probability over many rounds. The setting Fact 5.8 is also highly parametric, assuming that both the noise and initial state are i.i.d. Gaussian random variables. Corollary 5.7, on the other hand, makes no such assumptions, allowing an arbitrary initial state and conditionally sub-Gaussian noise variables.

An explicit comparison of the above bounds is difficult, but we can empirically compare the bounds by simulating a simple AR(1) model. We provide such a comparison in Figure 3

in Appendix D, which plots, for a fixed failure probability $\delta \in (0,1)$ the autoregressive guarantee from Corollary 5.7 against the corresponding guarantee provided by Fact 5.8. In Subfigure 3(a) we plot the bound from Fact 5.8 *without* providing a union bound correction. We thus emphasize that, as plotted, the boundary is only valid point-wise, and not for all $t \geq 1$ simultaneously or for arbitrary stopping times. In Subfigure 3(b), we make a union bound correction. Figure 3 indicates that Corollary 5.7 performs similarly to Fact 5.8 when specified to the scalar setting. We believe our bound may be preferable in application over that of Fact 5.8 as it is not only significantly more general, but it also inherently adapts to the variance of the observed autoregressive iterates.

6 A Self-Normalized, Multivariate Empirical Bernstein Inequality for Bounded Vectors

We construct a multivariate empirical Bernstein inequality, extending the Theorem 4 of Howard et al. [32] to higher dimensions. Empirical Bernstein-style bounds serve as a useful tool in common statistical tasks such as forming confidence sequences for estimating unknown means [58]. These bounds are of practical importance as they inherently adapt to the variance of a sequence of observations. If actual observations are tightly clustered, the resulting confidence bounds will be tighter. Likewise, if observations are well-dispersed, the resulting confidence set will be more conservative. To apply empirical Bernstein these bounds, a statistician must only know that the observations belong to a some bounded set.

To the best of our knowledge, we provide the first multivariate, self-normalized empirical Bernstein. Existing bounds either only hold in the scalar setting [58, 32], or do not normalize the quantity being estimated by the accumulated variance process (See, for instance, the work of Cutkosky [16] in the case of Hilbert space-valued variables). Providing confidence ellipsoids for mean estimation is desirable as it allows the confidence sets to reflect the "total amount of information" gathered in any given direction.

We now present the primary result of this section. In our result, we focus on the case where all observations have norm bounded above by $1/2$ for simplicity. This is mostly for theoretical convenience. While the more general setting where $(X_t)_{t \geq 1}$ belongs to some bounded, convex set is of interest, it can be readily analyzed by reducing to the case where observations lie in $\frac{1}{2}\mathbb{B}_d$[4].

For the remainder of this section, we adopt the notation ℓ_ρ^δ and L_ρ^δ instead of ℓ_ρ and L_ρ to explicitly make known the dependence on the confidence parameter δ. We make this dependence explicit as we will be union bounding in the sequel, and thus it will be useful to track the dependence.

---

[4]If $(X_t)_{t \geq 1}$ lies in some arbitrary convex, bounded set $K \subset \mathbb{R}^d$, we can first compute the outer John ellipsoid E of K, which is the minimal volume ellipsoid containing the convex set K [33]. In many settings, such as in the setting where K belongs to certain families of polytopes, there are computationally efficient algorithms that compute E [15, 53]. With E at hand, we can "recenter" our observations by defining a new sequence $(X_t')_{t \geq 0}$ by $X_t' := X_t - p$, where $p := \int_E x\,dx$ is the center of mass of E. We then have the equality $E - p = \frac{1}{2}A^{1/2}\mathbb{B}_d$, where A is some positive semi-definite matrix. We thus transform our observations into a final sequence $(X_t'')_{t \geq 0}$ defined by $X_t'' := A^{-1/2}(X_t - p)$, which lies almost surely in $\frac{1}{2}\mathbb{B}_d$.

Theorem 6.1. *Let $(X_t)_{t \geq 1}$ be a sequence of random vectors in $\mathbb{R}^d$ such that $\|X_t\| \leq 1/2$ almost surely, for all $t \geq 1$, and let $(\mathcal{F}_t)_{t \geq 0}$ be a filtration to which $(X_t)_{t \geq 1}$ is adapted. Then, the process $(S_t)_{t \geq 0}$ given by $S_t := \sum_{s=1}^{t}(X_s - \mathbb{E}_{s-1}X_s)$ is sub-$\psi_{E,1}$ with variance proxy $(V_t)_{t \geq 0}$ given by $V_t := \sum_{s=1}^{t}(X_s - \widehat{\mu}_{s-1})(X_s - \widehat{\mu}_{s-1})^\top$, where $\widehat{\mu}_t := t^{-1}\sum_{s=1}^{t}X_s$. Thus, by Theorem 4.1, for any fixed choice of parameters $\rho, \alpha, \delta, \beta, \epsilon, h$, we have*

$$\mathbb{P}\left(\exists t \geq 0 : \left\| (V_t \vee \rho)^{-1/2} S_t \right\| \geq \frac{\sqrt{\gamma_{\min}(V_t \vee \rho)}}{1 - \epsilon} \cdot (\psi_{E,1}^*)^{-1}\left(\frac{\alpha L_\rho^\delta(V_t)}{\gamma_{\min}(V_t \vee \rho)} \right) \right) \leq \delta.$$

In particular, since a sub-$\psi_{E,1}$ process is sub-$\psi_{G,1}$, this implies that, with probability at least $1 - \delta$, simultaneously for all $t \geq 0$,

$$\| (V_t \vee \rho)^{-1/2} S_t \| \leq \sqrt{2\alpha L_\rho^\delta(V_t)} + \frac{\alpha L_\rho^\delta(V_t)}{\gamma_{\min}(V_t \vee \rho)}.$$

We now compare the bound presented in Theorem 6.1 to existing empirical Bernstein-style results. In particular, our main point of comparison will be the following, scalar-valued bound from Howard et al. [32].

Proposition 6.2 ([**32, Theorem 4**]). *Suppose $(X_t)_{t \geq 1}$ satisfies $X_t \in [-1/2, 1/2]$ almost surely for all $t \geq 1$, and let $(S_t)_{t \geq 0}$, $(\widehat{\mu}_t)_{t \geq 0}$, and $(V_t)_{t \geq 0}$ be as in Theorem 6.1. For any choice of parameters α, δ, h, ρ, we have with probability at least $1 - \delta$, simultaneously for all $t \geq 1$,*

$$|S_t| \leq \sqrt{k_1^2(V_t \vee \rho)\ell_\rho^{2\delta}(V_t) + k_2^2\ell_\rho^{2\delta}(V_t)^2} + k_2\ell_\rho^{2\delta}(V_t),$$

where $k_1 := \frac{\alpha^{1/4} + \alpha^{-1/4}}{\sqrt{2}}$, $k_2 := \frac{\sqrt{\alpha}+1}{\sqrt{2}}$, and ℓ_ρ^δ is as given in Theorem 3.1.

Note that $\ell_\rho^{2\delta}$ appears as opposed to ℓ_ρ^δ in Proposition 6.2 due to an application of a a union bound in controlling both the upper and lower tail of S_t. In the case $d = 1$, Theorem 6.1 yields that, with probability at least $1 - \delta$, $|S_t| \leq \frac{1}{1-\epsilon}\left[\sqrt{2\alpha(V_t \vee \rho)L_\rho^\delta(V_t)} + L_\rho(V_t) \right]$. This serves as a sanity check, showing that up to small constants, the univariate bound presented in Theorem 6.1 is equivalent to that in Proposition 6.2. While one may expect the bound from Proposition 6.2 to be tighter for large values of V_t (as discussed in Section 3.1), this multiplicative gap can be made arbitrarily small by appropriately selecting tuning parameters.

7 Proofs of Main Results

In this section, we provide the proofs of what we view as the primary two results of this paper: Theorem 3.1 and Theorem 4.1. We additionally prove Corollary 4.6, which, while not a primary contribution of this work, has a proof that is similar in spirit to the other two results derived in this section. We start with the proof of Theorem 3.1, as the scalar bounds derived will play an integral role in the proof of Theorem 4.1

Proof of Theorem 3.1. First, observe that it suffices to show that, in the case $(S_t, V_t)_{t \geq 0}$ is sub-ψ and $V_t \geq 1, \forall t \geq 0$, we have

$$\mathbb{P}\left(\exists t \geq 0 : S_t \geq V_t \cdot (\psi^*)^{-1}\left(\frac{\alpha}{V_t}\ell_1(V_t)\right)\right) \leq \delta, \tag{5}$$

because, in the general case, we can consider the rescaled process $(S'_t, V'_t) := (S_t/\sqrt{\rho}, (V_t \vee \rho)/\rho)_{t \geq 0}$ and apply the concentration result from the case where $\rho = 1$. In more detail, clearly by construction $V'_t \geq 1$ for all $t \geq 1$, and by Proposition 2.3, we know $(S'_t, V'_t)_{t \geq 0}$ is sub-ψ_ρ, where we recall $\psi_\rho(\cdot) = \rho\psi(\cdot/\sqrt{\rho})$. Thus, noting that $(\psi^*_\rho)^{-1}(x) = \sqrt{\rho}(\psi^*)^{-1}(x/\rho)$ (Proposition A.2), we have

$$\delta \geq \mathbb{P}\left(\exists t \geq 0 : S'_t \geq V'_t \cdot (\psi^*_\rho)^{-1}\left(\frac{\alpha}{V'_t}\ell_1(V'_t)\right)\right)$$

$$= \mathbb{P}\left(\exists t \geq 0 : \frac{S_t}{\sqrt{\rho}} \geq \frac{V_t \vee \rho}{\rho} \cdot (\psi^*_\rho)^{-1}\left(\frac{\alpha\rho}{V_t \vee \rho}\ell_1\left(\frac{V_t \vee \rho}{\rho}\right)\right)\right)$$

$$= \mathbb{P}\left(\exists t \geq 0 : \frac{S_t}{\sqrt{\rho}} \geq \frac{V_t \vee \rho}{\rho}\sqrt{\rho} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{V_t \vee \rho}\ell_\rho(V_t)\right)\right)$$

$$= \mathbb{P}\left(\exists t \geq 0 : S_t \geq (V_t \vee \rho) \cdot (\psi^*)^{-1}\left(\frac{\alpha}{V_t \vee \rho}\ell_\rho(V_t)\right)\right),$$

which demonstrates the claimed bound in the theorem statement. Thus, going forward, we just prove the bound presented in (5).

For $k \in \mathbb{N}_0$, define the "intercept and slope" pair (x_k, m_k) by

$$x_k := \alpha^k(\psi^*)^{-1}\left(\frac{\log(h(k)/\delta)}{\alpha^k}\right), \qquad m_k := \alpha^k,$$

and define $g_k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$g_k(v) := x_k + \mathfrak{s}\left(\frac{x_k}{m_k}\right)(v - m_k),$$

where $\mathfrak{s}$ is the "slope transform" outlined in Appendix A. Since we have assumed $\lim_{\lambda \uparrow \lambda_{\max}} \psi'(\lambda) = \infty$, we can apply Lemma A.5 to obtain

$$\mathbb{P}\left(\exists t \geq 0 : S_t \geq g_k(V_t)\right) \leq \exp\left\{-m_k\psi^*\left(\frac{x_k}{m_k}\right)\right\} = \frac{\delta}{h(k)}.$$

Now, since $\mathfrak{s}(u) \leq u$ (Proposition A.6), observe that for $\alpha^k \leq v < \alpha^{k+1}$, we have

$$\min_{j \in \mathbb{N}_0} g_j(v) \leq g_k(v) = x_k + \mathfrak{s}\left(\frac{x_k}{m_k}\right)(v - m_k)$$

$$\leq x_k + \frac{x_k}{m_k}(v - m_k) = v\frac{x_k}{m_k} \tag{6}$$

$$= v \cdot (\psi^*)^{-1}\left(\frac{\log(h(k)/\delta)}{\alpha^k}\right)$$

$$\leq v \cdot (\psi^*)^{-1}\left(\frac{\alpha}{v}\log\left(\frac{h(\log_\alpha(v))}{\delta}\right)\right)$$

$$= v \cdot (\psi^*)^{-1}\left(\frac{\alpha}{v}\ell_1(v)\right),$$

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where the third inequality comes from the fact $k \le \log_\alpha(v)$, h is increasing, and $v \le \alpha^{k+1}$. Now, observe that we have, by a union bound

$$\mathbb{P}\left(\exists t \ge 0 : S_t \ge V_t \cdot (\psi^*)^{-1}\left(\frac{\alpha \ell_1(V_t)}{V_t}\right)\right) \le \mathbb{P}\left(\exists t \ge 0 : S_t \ge \min_{k \in \mathbb{N}_0} g_k(V_t)\right)$$

$$= \mathbb{P}\left(\bigcup_{k \in \mathbb{N}_0} \{\exists t \ge 0 : S_t \ge g_k(V_t)\}\right)$$

$$\le \sum_{k \in \mathbb{N}_0} \mathbb{P}(\exists t \ge 0 : S_t \ge g_k(V_t))$$

$$\le \delta \sum_{k \in \mathbb{N}_0} h(k)^{-1} \le \delta,$$

completing the proof. $\blacksquare$

We now go about proving Theorem 4.1. Before proving the theorem, we state a simple geometric lemma that will be needed in proving our result. In short, the following lemma states that a certain change of variables on $\mathbb{S}^{d-1}$ does not increase the distance between points of a covering to a significant degree. We prove the following in Appendix C.

Lemma 7.1. *Let K be a proper ϵ-cover of $\mathbb{S}^{d-1}$, and let $\pi : \mathbb{S}^{d-1} \to K$ be a projection mapping onto the cover K. Let T be a positive-definite matrix, and let $\kappa := \frac{\gamma_{\max}(T)}{\gamma_{\min}(T)}$ denote its condition number. Let $\pi_T : \mathbb{S}^{d-1} \to \mathbb{S}^{d-1}$ be defined as $\pi_T(\nu) := \frac{T^{1/2}\pi(\omega)}{\|T^{1/2}\pi(\omega)\|}$, where $\omega \in \mathbb{S}^{d-1}$ is the unique element satisfying*

$$\nu = \frac{T^{1/2}\omega}{\|T^{1/2}\omega\|}. \tag{7}$$

Then, for any $\nu \in \mathbb{S}^{d-1}$, we have

$$\|\nu - \pi_T(\nu)\| \le \sqrt{\kappa}\epsilon.$$

With the above lemma we can now prove the main result of the paper.

Proof of Theorem 4.1. Observe that if $(S_t, V_t)_{t \ge 0}$ is a sub-ψ process (in the sense of Definition 2.2), then so is $(S_t, V_t \vee \rho I_d)$, so it suffices to assume $V_t \succeq \rho I_d$ going forward.

For $j \in \mathbb{N}_0$, let K_j be a fixed, minimal proper $\frac{\epsilon}{\beta^j}$-cover of the unit sphere $\mathbb{S}^{d-1}$, and let $N_j := N(\mathbb{S}^{d-1}, \epsilon/\beta^j, \|\cdot\|)$. Let $\ell_\rho^{(j)} : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ be the function ℓ_ρ defined in Theorem 3.1 with δ set to the value δ_j defined by

$$\delta_j := \left(\frac{1 - \beta^{-1}}{\beta^j}\right) \delta/N_j.$$

That is, $\ell_\rho^{(j)}$ is the function given by

$$\ell_\rho^{(j)}(v) := \log\left(h\left(\log_\alpha\left(\frac{v \vee \rho}{\rho}\right)\right)\right) + \log\left(\frac{1}{\delta_j}\right)$$

$$= \log\left(h\left(\log_\alpha\left(\frac{v \vee \rho}{\rho}\right)\right)\right) + \log\left(\frac{\beta^j}{\delta(1 - \beta^{-1})}N_j\right).$$

Now, since $(S_t, V_t)_{t \geq 0}$ is an $\mathbb{R}^d$-valued sub-ψ process, by Definition 2.2, we know that, for any fixed $\nu \in \mathbb{S}^{d-1}$, $(\langle \nu, S_t \rangle, \langle \nu, V_t \nu \rangle)_{t \geq 0}$ is sub-ψ in the scalar sense of Definition 2.1. Hence, by applying Theorem 3.1, for any fixed $\nu \in \mathbb{S}^{d-1}$, we have

$$\mathbb{P}\left(\exists t \geq 0 : \langle \nu, S_t \rangle \geq \langle \nu, V_t \nu \rangle \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} \ell_\rho^{(j)}(\langle \nu, V_t \nu \rangle) \right) \right) \leq \left(\frac{1 - \beta^{-1}}{\beta^j} \right) \delta / N_j. \quad (8)$$

Noting that $\langle \nu, V_t \nu \rangle \leq \gamma_{\max}(V_t)$ for all $\nu \in \mathbb{S}^{d-1}$ and that $(\psi^*)^{-1}$ is an increasing function of its argument, we see that (8) still holds with $\ell_\rho^{(j)}(\langle \nu, V_t \nu \rangle)$ replaced by $\ell_\rho^{(j)}(\gamma_{\max}(V_t))$.

Now, for each $j \in \mathbb{N}_0$, define the "bad" event B_j as

$$B_j := \left\{ \exists t \geq 0, \nu \in K_j : \langle \nu, S_t \rangle \geq \langle \nu, V_t \nu \rangle \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} \ell_\rho^{(j)}(\gamma_{\max}(V_t)) \right) \right\}.$$

A straightforward union bound over the N_j elements of the cover K_j alongside (8) yields that $\mathbb{P}(B_j) \leq \frac{1 - \beta^{-1}}{\beta^j} \delta$. Defining now the global "bad" event B as

$$B := \left\{ \exists j \in \mathbb{N}_0, \exists \nu \in K_j, \exists t \geq 0 : \langle \nu, S_t \rangle \geq \langle \nu, V_t \nu \rangle \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} \ell_\rho^{(j)}(\gamma_{\max}(V_t)) \right) \right\} = \bigcup_{j \in \mathbb{N}_0} B_j.$$

An additional straightforward union bound over indices $j \in \mathbb{N}_0$ yields

$$\mathbb{P}(B) = \mathbb{P}\left(\bigcup_{j \in \mathbb{N}_0} B_j \right) \leq \sum_{j \in \mathbb{N}_0} \mathbb{P}(B_j) \leq (1 - \beta^{-1}) \delta \sum_{j \in \mathbb{N}_0} \beta^{-j} = \delta.$$

Now, for $j \in \mathbb{N}_0$ and $t \geq 0$, let $\pi_t^{(j)} := \pi_{V_t}$ be the projection mapping from $\mathbb{S}^{d-1}$ onto the finite set $K_j(t) := \left\{ V_t^{1/2} \nu / \| V_t^{1/2} \nu \| : \nu \in K_j \right\} \subset \mathbb{S}^{d-1}$, as in Lemma 7.1. Note that while $K_j(t)$ is a random subset of the unit sphere (through its dependence on the "accumulated variance" operator V_t at time t), the underlying $\frac{\epsilon}{\beta^j}$-cover K_j of $\mathbb{S}^{d-1}$ is fixed. Further, for $j \in \mathbb{N}_0$ and $t \geq 0$, define the event $E_j(t)$ by

$$E_j(t) := \left\{ \beta^j \leq \sqrt{\kappa(V_t)} < \beta^{j+1} \right\}.$$

On the event $E_j(t)$, for any $j \in \mathbb{N}_0$ and $t \geq 0$, we have

$$\left\| V_t^{-1/2} S_t \right\| = \sup_{\omega \in \mathbb{S}^{d-1}} \left\langle \omega, V_t^{-1/2} S_t \right\rangle = \sup_{\omega \in \mathbb{S}^{d-1}} \left\{ \left\langle \omega - \pi_t^{(j+1)}(\omega), V_t^{-1/2} S_t \right\rangle + \left\langle \pi_t^{(j+1)}(\omega), V_t^{-1/2} S_t \right\rangle \right\}$$

$$\leq \sup_{\omega \in \mathbb{S}^{d-1}} \left\| \omega - \pi_t^{(j+1)}(\omega) \right\| \cdot \left\| V_t^{-1/2} S_t \right\| + \sup_{\omega \in K_{j+1}(t)} \left\langle \omega, V_t^{-1/2} S_t \right\rangle$$

$$\leq \frac{\epsilon}{\beta^{j+1}} \sqrt{\kappa(V_t)} \left\| V_t^{-1/2} S_t \right\| + \sup_{\nu \in K_{j+1}} \left\langle \frac{V_t^{1/2} \nu}{\| V_t^{1/2} \nu \|}, V_t^{-1/2} S_t \right\rangle$$

$$\leq \epsilon \left\| V_t^{-1/2} S_t \right\| + \sup_{\nu \in K_{j+1}} \frac{\langle \nu, S_t \rangle}{\sqrt{\langle \nu, V_t \nu \rangle}}.$$

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In the above, the first equality comes from the variational representation of the norm $\|\cdot\|$ and the second equality comes from adding and subtracting $\left\langle \pi_t^{(j+1)}(\omega), V_t^{-1/2} S_t \right\rangle$. Further, the first inequality comes from splitting the supremum and applying Cauchy-Schwarz to the first term, the second inequality comes from applying Lemma 7.1 to $\left\| \omega - \pi_t^{(j+1)}(\omega) \right\|$ and applying the definition of $K_{j+1}(t)$, and the final inequality comes from simplifying the second term and from observing that, on the event $E_j(t)$, $\sqrt{\kappa(V_t)} < \beta^{j+1}$.

Further, observe that, on the event $E_j(t)$, we have the inequality

$$
\begin{aligned}
\ell_\rho^{(j+1)}(\gamma_{\max}(V_t)) &= \log\left(h\left(\log_\alpha\left(\frac{\gamma_{\max}(V_t) \vee \rho}{\rho} \right) \right) \right) + \log\left(\frac{1}{\delta(1-\beta^{-1})} \right) + \log\left(N_j \beta^{j+1} \right) \\
&\leq \log\left(h\left(\log_\alpha\left(\frac{\gamma_{\max}(V_t) \vee \rho}{\rho} \right) \right) \right) + \log\left(\frac{1}{\delta(1-\beta^{-1})} \right) \\
&\quad + \log\left(\beta\sqrt{\kappa(V_t)} N_{d-1}\left(\frac{\epsilon}{\beta\sqrt{\kappa(V_t)}} \right) \right) \\
&= L_\rho(V_t).
\end{aligned}
$$

In the above, the inequality follows from observing that $\beta^j \leq \sqrt{\kappa(V_t)}$. From this, rearranging, we see that, for any $j \in \mathbb{N_0}$ and $t \geq 0$, on the event $E_j(t) \cap B^c$ we have

$$
\begin{aligned}
\left\| V_t^{-1/2} S_t \right\| &\leq \frac{1}{1-\epsilon} \sup_{\nu \in K_{j+1}} \frac{\langle \nu, S_t \rangle}{\sqrt{\langle \nu, V_t \nu \rangle}} \\
&\leq \frac{1}{1-\epsilon} \sup_{\nu \in K_{j+1}} \sqrt{\langle \nu, V_t \nu \rangle} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} \ell_\rho(\gamma_{\max}(V_t)) \right) \\
&\leq \frac{1}{1-\epsilon} \sup_{\nu \in K_{j+1}} \sqrt{\langle \nu, V_t \nu \rangle} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} L_\rho(V_t) \right) \\
&\leq \frac{1}{1-\epsilon} \sup_{\nu \in \mathbb{S}^{d-1}} \sqrt{\langle \nu, V_t \nu \rangle} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} L_\rho(V_t) \right).
\end{aligned}
$$

If $(j_t)_{t\in\mathbb{N_0}}$ is any sequence of natural numbers, and we define $G^{(j_t)} := \bigcap_{t\geq 0} \{E_{j_t}(t) \cap B^c\}$, it is clear that, on the event $G^{(j_t)}$, the inequality

$$
\left\| V_t^{-1/2} S_t \right\| \leq \frac{1}{1-\epsilon} \sup_{\nu \in \mathbb{S}^{d-1}} \sqrt{\langle \nu, V_t \nu \rangle} \cdot (\psi^*)^{-1}\left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} L_\rho(V_t) \right) \tag{9}
$$

holds simultaneously for all $t \geq 0$. Noting that we have the identity $B^c = \biguplus_{(j_t)_{t\in\mathbb{N_0}}} G^{(j_t)}$ yields that (9) actually holds simultaneously for all $t \geq 0$ on the event B^c. What we have done in the above is break the "good" event B^c into geometric buckets based on the condition number at each time, and then noted that the regardless of the realized sequence of condition numbers $(\kappa(V_t))_{t\geq 0}$, the target inequality holds.

This proves the claim for arbitrary CGF-like functions ψ, which is presented following Theorem 4.1. Now, if we further assume ψ is super-Gaussian, on the event B^c defined above,

we have

$$\left\| V_t^{-1/2} S_t \right\| \leq \frac{1}{1-\epsilon} \sup_{\nu \in \mathbb{S}^{d-1}} \sqrt{\langle \nu, V_t \nu \rangle} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\langle \nu, V_t \nu \rangle} L_\rho(V_t) \right)$$

$$= \frac{1}{1-\epsilon} \sup_{x \in [\gamma_{\min}(V_t), \gamma_{\max}(V_t)]} \sqrt{x} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{x} L_\rho(V_t) \right).$$

Now, by Lemma A.3, we know the assumption that ψ is a super-Gaussian CGF-like function implies that ψ^* is a sub-Gaussian CGF-like function. Moreover by the same proposition, we see that $\psi^*(C\cdot)$ is a sub-Gaussian CGF-like function for any positive $C > 0$. Consequently, by Proposition A.3, we see that $(\psi^*)^{-1}(Cu)/\sqrt{u}$ is an increasing function of u, and thus by making the change of variable $x := \frac{1}{u}$, that $\sqrt{x}(\psi^*)^{-1}\left(\frac{C}{x}\right)$ a decreasing function of x. Thus, we have that, on the event B^c (which, we recall, occurs with probability at least $1 - \delta$)

$$\left\| V_t^{-1/2} S_t \right\| \leq \frac{1}{1-\epsilon} \sup_{x \in [\gamma_{\min}(V_t), \gamma_{\max}(V_t)]} \sqrt{x} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{x} L_\rho(V_t) \right) \leq \frac{\sqrt{\gamma_{\min}(V_t)}}{1-\epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\gamma_{\min}(V_t)} L_\rho(V_t) \right)$$

simultaneously for all $t \geq 0$, proving the desired result. A symmetric argument holds in the case that the CGF-like function ψ is instead sub-Gaussian, with $\gamma_{\max}(V_t)$ replacing $\gamma_{\min}(V_t)$ in the final inequality.

∎

Lastly, we prove Corollary 4.6, which in turn can be used to derive Corollary 3.2. While we do not consider this corollary a primary contribution of our work, we include the proof in this section due to its closeness (in spirit) to the previous two proofs.

Proof of Corollary 4.6. Recalling that $(\psi_N^*)^{-1}(u) = \sqrt{2u}$, the assumption that $\psi(\lambda) \sim \frac{\lambda^2}{2}$ implies there, for any $\eta > 0$, there exists an $\overline{u} \in \mathbb{R}_{>0}$ such that

$$(\psi^*)^{-1}(u) \leq (1+\eta)(\psi_N^*)^{-1}(u) = (1+\eta)\sqrt{2u}$$

for all $u \in [0, \overline{u}]$. Let $(\alpha_n)_{n \geq 1}, (s_n)_{n \geq 1}, (\epsilon_n)_{n \geq 1}$, and $(\beta_n)_{n \geq 1}$ be such that $\alpha_n, s_n, \beta_n \downarrow 1$ and $\epsilon_n \downarrow 0$ monotonically and let $(\delta_n)_{n \geq 1}$ be such that (a) $\delta_n \downarrow 0$ monotonically and (b) $\sum_{n=1}^\infty \delta_n < \infty$. Define the sequence of functions $h_n : \mathbb{N}_0 \to \mathbb{R}_{\geq 0}$ by $h_n(m) := \zeta(s_n)(1+m)^{s_n}$. Let T_n be the (almost surely finite) random time given by

$$T_n := \inf \left\{ t \geq 0 : \frac{\alpha}{\gamma_{\min}(V_{t'})} L_1(V_{t'}) \leq \overline{u} \ \forall t' \geq t, \ \text{and} \right.$$

$$\left. \log \left(\frac{C_d \zeta(s_n)}{\delta_n (\log(\alpha_n))^{s_n} (1 - \beta_n^{-1})} \right) + d \log \left(\frac{3\beta_n}{\epsilon_n} \right) \leq \frac{s_n}{n} \log \log(\gamma_{\max}(V_t)) \right\}.$$

Theorem 4.1 instantiated with the covering number bound in Lemma 4.2 implies that, with

probability at least $1 - \delta_n$, simultaneously for all $t \geq T_n$, we have

$$\left\| V_t^{-1/2} S_t \right\| \leq \frac{\sqrt{\gamma_{\min}(V_t)}}{1 - \epsilon_n} \cdot (\psi^*)^{-1} \left(\frac{\alpha_n}{\gamma_{\min}(V_t)} L_1(V_t) \right)$$

$$\leq \frac{1 + \eta}{1 - \epsilon_n} \sqrt{2\alpha_n \left[s_n \log \log(V_t) + \log \left(\frac{C_d \zeta(s_n)}{\delta_n (\log(\alpha_n))^{s_n} (1 - \beta_n^{-1})} \right) + d \log \left(\frac{3\beta_n \sqrt{\kappa(V_t)}}{\epsilon_n} \right) \right]}$$

$$\leq \frac{1 + \eta}{1 - \epsilon_n} \sqrt{2\alpha_n s_n \left(1 + \frac{1}{n} \right) \log \log(V_t) + \alpha_n d \log \kappa(V_t)}.$$

Thus, for $n \geq 1$, define the event A_n by

$$A_n = \left\{ \exists t \geq T_n : \left\| V_t^{-1/2} S_t \right\| \geq \frac{1 + \eta}{1 - \epsilon_n} \sqrt{2\alpha_n s_n \left(1 + \frac{1}{n} \right) \log \log(V_t) + \alpha_n d \log \kappa(V_t)} \right\},$$

and observe that by the above argument $\mathbb{P}(A_n) \leq \delta_n$. Note that, for arbitrary $\gamma > 1$, we have

$$A_\gamma := \left\{ \|V_t^{-1/2} S_t\| > (1 + \eta) \sqrt{\gamma \left[2 \log \log(V_t) + d \log \kappa(V_t) \right]} \ \text{i.o.} \right\} \subset \limsup_{n \to \infty} A_n := \bigcap_{n \geq 1} \bigcup_{k \geq n} A_k,$$

where i.o. denotes an event occurring infinitely often. By the first Borel-Cantelli lemma (see Durrett [26], Chapter 2) we have

$$\mathbb{P}(A_\gamma) \leq \mathbb{P} \left(\bigcap_{n \geq 1} \bigcup_{k \geq n} A_k \right) = 0,$$

since $\sum_{n=1}^\infty \mathbb{P}(A_n) \leq \sum_{n=1}^\infty \delta_n < \infty$. Thus, with probability 1, we have

$$\limsup_{t \to \infty} \frac{S_t}{(1 + \eta) \sqrt{\gamma \left[2 \log \log(V_t) + d \log \kappa(V_t) \right]}} \leq 1,$$

but since $\eta > 0$ and $\gamma > 1$ where arbitrary, the result follows. $\blacksquare$

8 Conclusion and Discussion

Self-normalized quantities arise naturally in a variety of high-dimensional statistical tasks, with online learning [3, 1, 61, 12, 14], time series analysis [8, 50], and hypothesis testing [51, 52, 48, 58] being several notable examples. Despite their crucial role in common statistical tasks, very little has been explored in terms of self-normalized concentration outside of the sub-Gaussian setting. In this paper, we present a time-uniform, self-normalized concentration for sub-ψ processes, i.e. processes whose increments, roughly, have cumulant generating function bounded by ψ. Our results are closed form, have small constants, and

have parameters that can be fine-tuned for a statistician's desired application. Moreover, with our bounds, we can establish an asymptotic law of the iterated logarithm for vector-valued processes that recovers the law of iterated logarithm for scalar sub-ψ processes first established by Howard et al. [32].

Along with our primary result on the self-normalized concentration of vector-valued processes, we make variety of additional contributions. En route to proving Theorem 4.1, we prove a non-asymptotic law of the iterated for sub-ψ processes, generalizing the results of Howard et al. [32] beyond just the sub-Gamma setting. Likewise, we demonstrate how to leverage our self-normalized inequalities in several practical statistical settings. In particular, we derive non-asymptotically valid confidence ellipsoids for online linear regression, describe how to construct confidence sets for vector autoregressive models, and prove a multivariate empirical Bernstein inequality. There are undoubtedly many more settings in which our bounds can be applied, and we leave the exploration of these applications for interesting future work.

While the results presented in this paper are quite general, there are still many interesting questions about self-normalized concentration to be answered. As a first example, existing results on the self-normalized concentration of sub-Gaussian random vectors yield a bound that is proportional to $O\left(\sqrt{\log \det(V_t)}\right)$ [21, 22, 2]. This is in contrast to the results discussed in this work, which provide bounds of the form $O\left(\sqrt{\log \log \gamma_{\max}(V_t) + d \log \kappa(V_t)}\right)$. As discussed in Section 4, neither form of bound uniformly dominates the other. In particular, when V_t is well-conditioned, our concentration results may be preferable, but for poorly-conditioned V_t, determinant rate bounds may be desirable. A major open question is whether determinant rate bounds can be obtained for general sub-ψ processes, or if the determinant rate is just attainable in the sub-Gaussian setting. The techniques discussed in this paper do not seem directly applicable to this setting, and so we thus leave obtaining determinant rate bounds as compelling future work.

This work demonstrates that simple, closed-form bounds on self-normalized processes can be established under very general distributional assumptions. While existing works consider a setting in which the increments of processes are sub-Gaussian, concentration of measure should not be viewed as a "one size fits all" phenomenon. For instance, the noise observed in taking real-world may not be sub-Gaussian, but rather perhaps sub-Exponential, sub-Gamma, or even heavy tailed. Overall, our bounds provide a means by which the statistician can properly calibrate confidence in these more delicate settings.

9 Acknowledgements

References

[1] Yasin Abbasi-Yadkori. *Online learning for linearly parametrized control problems*. PhD thesis, University of Alberta, 2013.

[2] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Improved algorithms for linear stochastic bandits. *Advances in Neural Information Processing Systems*, 24, 2011.

[3] Yasin Abbasi-Yadkori, Dávid Pál, and Csaba Szepesvári. Online least squares estimation with self-normalized processes: An application to bandit problems. *arXiv preprint arXiv:1102.2670*, 2011.

[4] Anish Agarwal, Muhammad Jehangir Amjad, Devavrat Shah, and Dennis Shen. Model agnostic time series analysis via matrix estimation. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, 2(3):1–39, 2018.

[5] Rudolf Ahlswede and Andreas Winter. Strong converse for identification via quantum channels. *IEEE Transactions on Information Theory*, 48(3):569–579, 2002.

[6] Kazuoki Azuma. Weighted sums of certain dependent random variables. *Tohoku Mathematical Journal, Second Series*, 19(3):357–367, 1967.

[7] George Bennett. Probability inequalities for the sum of independent random variables. *Journal of the American Statistical Association*, 57(297):33–45, 1962.

[8] Bernard Bercu and Abderrahmen Touati. Exponential inequalities for self-normalized martingales with applications. *The Annals of Applied Probability*, 18(5):1848–1869, 2008.

[9] Bernard Bercu and Taieb Touati. New insights on concentration inequalities for self-normalized martingales. *Electronic Communications in Probability*, 24:1 – 12, 2019.

[10] D Blackwell. Large deviations for martingales. *Festschrift for Lucien Le Cam: Research Papers in Probability and Statistics*, pages 89–91, 1997.

[11] Stéphane Boucheron, Gábor Lugosi, and Pascal Massart. *Concentration inequalities: a nonasymptotic theory of independence*. Oxford university press, 2013.

[12] Sayak Ray Chowdhury and Aditya Gopalan. On kernelized multi-armed bandits. In *International Conference on Machine Learning*, pages 844–853. PMLR, 2017.

[13] Demetres Christofides and Klas Markström. Expansion properties of random Cayley graphs and vertex transitive graphs via matrix martingales. *Random Structures & Algorithms*, 32(1):88–100, 2008.

[14] Ben Chugg, Hongjian Wang, and Aaditya Ramdas. A unified recipe for deriving (time-uniform) PAC-Bayes bounds. *arXiv preprint arXiv:2302.03421*, 2023.

[15] Michael B Cohen, Ben Cousins, Yin Tat Lee, and Xin Yang. A near-optimal algorithm for approximating the John ellipsoid. In *Conference on Learning Theory*, pages 849–873. PMLR, 2019.

[16] Ashok Cutkosky. Combining online learning guarantees. In *Conference on Learning Theory*, pages 895–913. PMLR, 2019.

[17] DA Darling and Herbert Robbins. Iterated logarithm inequalities. *Proceedings of the National Academy of Sciences*, 57(5):1188–1192, 1967.

[18] DA Darling and Herbert Robbins. Some further remarks on inequalities for sample sums. *Proceedings of the National Academy of Sciences*, 60(4):1175–1182, 1968.

[19] Serge Darolles, Christian Gourieroux, and Joann Jasiak. Structural Laplace transform and compound autoregressive models. *Journal of Time Series Analysis*, 27(4):477–503, 2006.

[20] Victor de la Peña, Michael J Klass, and Tze Leung Lai. Self-normalized processes: exponential inequalities, moment bounds and iterated logarithm laws. *The Annals of Probability*, 32(3):1902 – 1933, 2004.

[21] Victor de la Peña, Michael J Klass, and Tze Leung Lai. Pseudo-maximization and self-normalized processes. *Probability Surveys*, 4:172 – 192, 2007.

[22] Victor de la Peña, Michael J Klass, and Tze Leung Lai. Theory and applications of multivariate self-normalized processes. *Stochastic Processes and their Applications*, 119 (12):4210–4227, 2009.

[23] Victor de la Peña, Tze Leung Lai, and Qi-Man Shao. *Self-normalized Processes: Limit Theory and Statistical Applications*. Springer, 2009.

[24] Joseph L Doob. Regularity properties of certain families of chance variables. *Transactions of the American Mathematical Society*, 47(3):455–486, 1940.

[25] Audrey Durand, Odalric-Ambrym Maillard, and Joelle Pineau. Streaming kernel regression with provably adaptive mean, variance, and regularization. *The Journal of Machine Learning Research*, 19(1):650–683, 2018.

[26] Rick Durrett. *Probability: Theory and Examples*, volume 49. Cambridge university press, 2019.

[27] Xiequan Fan, Ion Grama, and Quansheng Liu. Exponential inequalities for martingales with applications. *Electronic Journal of Probability*, 20(none):1 – 22, 2015.

[28] David A Freedman. On tail probabilities for martingales. *The Annals of Probability*, pages 100–118, 1975.

[29] James D Hamilton. *Time Series Analysis*. Princeton university press, 2020.

[30] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.

[31] Steven R Howard, Aaditya Ramdas, Jon McAuliffe, and Jasjeet Sekhon. Time-uniform Chernoff bounds via nonnegative supermartingales. *Probability Surveys*, 17:257–317, 2020.

[32] Steven R Howard, Aaditya Ramdas, Jon McAuliffe, and Jasjeet Sekhon. Time-uniform, nonparametric, nonasymptotic confidence sequences. *The Annals of Statistics*, 49(2): 1055–1080, 2021.

[33] Fritz John. Extremum problems with inequalities as subsidiary conditions. *Traces and Emergence of Nonlinear Programming*, pages 197–215, 2014.

[34] Kwang-Sung Jun and Francesco Orabona. Parameter-free online convex optimization with sub-exponential noise. In *Conference on Learning Theory*, pages 1802–1823. PMLR, 2019.

[35] Emilie Kaufmann, Olivier Cappé, and Aurélien Garivier. On the complexity of best arm identification in multi-armed bandit models. *Journal of Machine Learning Research*, 17: 1–42, 2016.

[36] Robert W Keener. *Theoretical Statistics: Topics for a Core Course*. Springer, 2010.

[37] Akshay Krishnamurthy, Zhiwei Steven Wu, and Vasilis Syrgkanis. Semiparametric contextual bandits. In *International Conference on Machine Learning*, pages 2776–2785. PMLR, 2018.

[38] Arun K Kuchibhotla and Rohit K Patra. On least squares estimation under heteroscedastic and heavy-tailed errors. *The Annals of Statistics*, 50(1):277–302, 2022.

[39] Arun Kumar Kuchibhotla and Abhishek Chakrabortty. Moving beyond sub-Gaussianity in high-dimensional statistics: applications in covariance estimation and linear regression. *Information and Inference: A Journal of the IMA*, 11(4):1389–1456, 2022.

[40] Tze Leung Lai and Herbert Robbins. Consistency and asymptotic efficiency of slope estimates in stochastic approximation schemes. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 56(3):329–360, 1981.

[41] Tze Leung Lai and Ching Zong Wei. Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems. *The Annals of Statistics*, 10(1):154–166, 1982.

[42] Tor Lattimore and Csaba Szepesvári. *Bandit Algorithms*. Cambridge University Press, 2020.

[43] Xiaoyu Li and Francesco Orabona. A high probability analysis of adaptive SGD with momentum. *arXiv preprint arXiv:2007.14294*, 2020.

[44] Tudor Manole and Aaditya Ramdas. Martingale methods for sequential estimation of convex functionals and divergences. *IEEE Transactions on Information Theory*, 2023.

[45] Maria Pacurar. Autoregressive conditional duration models in finance: a survey of the theoretical and empirical literature. *Journal of Economic Surveys*, 22(4):711–751, 2008.

[46] Iosif Pinelis. An approach to inequalities for the distributions of infinite-dimensional martingales. *Probability in Banach Spaces*, 8:128–134, 1992.

[47] Iosif Pinelis. Optimum bounds for the distributions of martingales in Banach paces. *The Annals of Probability*, pages 1679–1706, 1994.

[48] Aleksandr Podkopaev, Patrick Blöbaum, Shiva Kasiviswanathan, and Aaditya Ramdas. Sequential kernelized independence testing. In *International Conference on Machine Learning*, pages 27957–27993. PMLR, 2023.

[49] Henry Scheffe. *The Analysis of Variance*, volume 72. John Wiley & Sons, 1999.

[50] Xiaofeng Shao. Self-normalization for time series: a review of recent developments. *Journal of the American Statistical Association*, 110(512):1797–1817, 2015.

[51] Shubhanshu Shekhar and Aaditya Ramdas. Nonparametric two-sample testing by betting. *IEEE Transactions on Information Theory*, 2023.

[52] Shubhanshu Shekhar, Ilmun Kim, and Aaditya Ramdas. A permutation-free kernel two-sample test. *Advances in Neural Information Processing Systems*, 35:18168–18180, 2022.

[53] Zhao Song, Xin Yang, Yuanyuan Yang, and Tianyi Zhou. Faster algorithm for structured John ellipsoid computation. *arXiv preprint arXiv:2211.14407*, 2022.

[54] Joel Tropp. Freedman's inequality for matrix martingales. *Electronic Communications in Probability*, 16, 2011.

[55] Joel A Tropp. User-friendly tail bounds for sums of random matrices. *Foundations of Computational Mathematics*, 12:389–434, 2012.

[56] Jean Ville. Etude critique de la notion de collectif. *Bull. Amer. Math. Soc*, 45(11):824, 1939.

[57] Martin J Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, volume 48. Cambridge university press, 2019.

[58] Ian Waudby-Smith and Aaditya Ramdas. Estimating means of bounded random variables by betting. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, 2023.

[59] Justin Whitehouse, Aaditya Ramdas, Steven Z Wu, and Ryan M Rogers. Brownian noise reduction: Maximizing privacy subject to accuracy constraints. *Advances in Neural Information Processing Systems*, 35:11217–11228, 2022.

[60] Justin Whitehouse, Aaditya Ramdas, Ryan Rogers, and Steven Wu. Fully-adaptive composition in differential privacy. In *International Conference on Machine Learning*, pages 36990–37007. PMLR, 2023.

[61] Justin Whitehouse, Zhiwei Steven Wu, and Aaditya Ramdas. On the sublinear regret of GP-UCB. *Advances in Neural Information Processing Systems*, 2023.

A   Properties of CGF-like Functions

The *cumulant generating function* (or *CGF*) of a random variable plays an integral role in understanding concentration of measure phenomena, such as through the classical Chernoff style of argument [31, 11]. Suppose X is a random variable such that $\mathbb{E}X = 0$, $\mathbb{E}e^{\lambda X} < \infty$ for all $\lambda \in [0, \lambda_{\max})$, and $\lim_{\lambda \uparrow \lambda_{\max}} \mathbb{E}e^{\lambda X} = \infty$. The cumulant generating function of X, which can be thought of as "compressing" all of the moments of X into a single function, is the mapping $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ given by $\psi(\lambda) := \log \mathbb{E}e^{\lambda X}$.

In this appendix, we study properties of *cumulant generating function-like* (or *CGF-like*) functions, which are functions that may not be the CGF of any random variable, but display similar analytic properties to CGFs. If $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is the CGF of a random variable, straightforward calculation yields that $\psi(0) = \psi'(0) = 0$, $\psi''(\lambda) > 0$, and ψ is strictly convex. As such, we say a twice continuously differentiable function $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is CGF-like if it obeys these aforementioned properties. We study various properties of CGF-like functions in the sequel, as these properties form the foundation of our results studying the self-normalized concentration of sub-ψ processes.

Proposition A.1. *Suppose $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is CGF-like. Then convex conjugate $\psi^* : [0, u_{\max}) \to \mathbb{R}_{\geq 0}$ defined by $\psi^*(u) := \sup_{\lambda \in [0, \lambda_{\max})} \lambda u - \psi(u)$, is also CGF-like, where $u_{\max} := \sup_{\lambda \in [0, \lambda_{\max})} \psi'(\lambda)$.*

Proof. Clearly ψ^* is convex and twice continuously-differentiable. Next, observe that

$$\psi^*(0) = \sup_{\lambda \in [0, \lambda_{\max})} \{-\psi(\lambda)\} = \psi(0) = 0.$$

Further, using the fact that $(\psi^*)' = (\psi')^{-1}$, we have that

$$(\psi^*)'(0) = (\psi')^{-1}(0) = 0.$$

Lastly, we have that

$$(\psi^*)''(0) = ((\psi')^{-1})'(0) = \frac{1}{\psi''((\psi')^{-1}(0))} = \frac{1}{\psi''(0)} > 0.$$

Thus, ψ^* is also CGF-like. ∎

If $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is CGF-like, then for any $\rho > 0$, the "rescaled" function $\psi_\rho : [0, \sqrt{\rho}\lambda_{\max}) \to \mathbb{R}_{\geq 0}$ given by $\psi_\rho(\lambda) := \rho\psi(\lambda/\sqrt{\rho})$ is also CGF-like. These rescaled CGF-like functions arise naturally in studying processes that have been re-normalized to have $V_t \succeq \mathrm{id}_H$ for all $t \geq 0$. These rescaled functions ψ_ρ exhibit the following properties.

Proposition A.2. *Let* $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ *be CGF-like, and let* $\psi_\rho : [0, \sqrt{\rho}\lambda_{\max}) \to \mathbb{R}_{\geq 0}$ *be as above. The following hold.*

1. ψ_ρ *is a CGF-like function.*

2. $\psi_\rho^*(u) = \rho\psi^*(u/\sqrt{\rho})$.

3. $(\psi_\rho^*)^{-1}(x) = \sqrt{\rho}(\psi^*)^{-1}\left(\frac{x}{\rho}\right)$.

Proof. The validity of the first claim follows immediately by the definition of a CGF-like function.

To see the validity of the second claim, note that

$$\psi_\rho^*(u) = \sup_{\lambda \in [0, \sqrt{\rho}\lambda_{\max})} \left\{ u\lambda - \rho\psi\left(\frac{\lambda}{\sqrt{\rho}}\right) \right\}.$$

Differentiating the inner expression on the right-hand side and setting equal to zero furnishes that the supremum is obtained at $\lambda = \sqrt{\rho}(\psi')^{-1}(u/\sqrt{\rho})$. Plugging this back into the above expression yields

$$
\begin{aligned}
\psi_\rho^*(u) &= \sqrt{\rho}u(\psi')^{-1}\left(\frac{u}{\sqrt{\rho}}\right) - \rho\psi\left((\psi')^{-1}\left(\frac{u}{\sqrt{\rho}}\right)\right) \\
&= \rho\left[\frac{u}{\sqrt{\rho}}(\psi')^{-1}\left(\frac{u}{\sqrt{\rho}}\right) - \psi\left((\psi')^{-1}\left(\frac{u}{\sqrt{\rho}}\right)\right)\right] \\
&= \rho\psi^*\left(\frac{u}{\sqrt{\rho}}\right),
\end{aligned}
$$

which proves the second item.

Laslty, the third item can be readily checked as

$$\psi_\rho^*\left(\sqrt{\rho}(\psi^*)^{-1}\left(\frac{x}{\rho}\right)\right) = \rho(\psi^*)\left(\frac{\sqrt{\rho}}{\sqrt{\rho}}(\psi^*)^{-1}\left(\frac{x}{\rho}\right)\right) = \rho\frac{x}{\rho} = x.$$

Applying $(\psi_\rho^*)^{-1}$ to both sides thus yields the desired result. ∎

Throughout our work, we are especially interested in studying sub-ψ processes whose increments exhibit tail behavior that is either "heavier" or "lighter" than that of a Gaussian random-variable. More concretely, we study processes where ψ is a *super-Gaussian* (respectively *sub-Gaussian*) CGF-like function, i.e. a CGF-like function where $\frac{\psi(\lambda)}{\lambda^2}$ is a non-decreasing (respectively non-increasing) function of λ. In words, a CGF-like function ψ is super-Gaussian (or sub-Gaussian) if it increases more rapidly (less rapdily) than the CGF of a standard normal random variable. We focus on super-Gaussian CGF-like functions in the sequel, but exactly analogous results hold for sub-Gaussian CGF-like functions. Super-Gaussian CGF-like functions enjoy a number of convenient properties and equivalent definitions, which we enumerate below.

Proposition A.3. *Suppose $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is a CGF-like function. The following hold.*

1. *ψ is super-Gaussian if and only if $\psi'(\lambda) \geq \frac{2\psi(\lambda)}{\lambda}$.*

2. *If ψ is super-Gaussian, then so is $\varphi := a\psi(b\cdot) : [0, \lambda_{\max}/b) \to \mathbb{R}_{\geq 0}$ for any $a, b > 0$.*

3. *If ψ is super-Gaussian, then $\frac{\psi^{-1}(x)}{\sqrt{x}}$ is a decreasing function of $x \in [0, \infty)$.*

4. *ψ is super-Gaussian if and only if its convex conjugate ψ^* is sub-Gaussian.*

Proof.   1. Differentiating via the product rule yields

$$\left(\frac{\psi(\lambda)}{\lambda^2}\right)' = \frac{\psi'(\lambda)}{\lambda^2} - \frac{2\psi(\lambda)}{\lambda^3}.$$

Consequently, we have

$$\left(\frac{\psi(\lambda)}{\lambda^2}\right)' \geq 0 \quad \Leftrightarrow \quad \psi'(\lambda) \geq \frac{2\psi(\lambda)}{\lambda},$$

proving the desired result.

2. This result follows from the equivalent condition presented in the first part of the proposition. In particular, observe that we have

$$\varphi'(\lambda) = ab\psi'(b\lambda) \geq 2ab\frac{\psi(b\lambda)}{b\lambda} = \frac{2\varphi(\lambda)}{\lambda},$$

proving the desired result.

3. Straightforward calculus yields

$$\left(\frac{\psi^{-1}(x)}{\sqrt{x}}\right)' = \frac{(\psi^{-1})'(x)}{\sqrt{x}} - \frac{1}{2}\frac{\psi^{-1}(x)}{x^{3/2}} = \frac{1}{\sqrt{x}\psi'(\psi^{-1}(x))} - \frac{1}{2}\frac{\psi^{-1}(x)}{x^{3/2}}.$$

Next, the assumption of ψ being super-Gaussian yields

$$\psi'(\psi^{-1}(x)) \geq \frac{2\psi(\psi^{-1}(x))}{\psi^{-1}(x)} = \frac{2x}{\psi^{-1}(x)}.$$

Combining these two panels furnishes

$$\left(\frac{\psi^{-1}(x)}{\sqrt{x}}\right)' = \frac{1}{\sqrt{x}\psi'(\psi^{-1}(x))} - \frac{1}{2}\frac{\psi^{-1}(x)}{x^{3/2}} \leq \frac{1}{2}\frac{\psi^{-1}(x)}{x^{3/2}} - \frac{1}{2}\frac{\psi^{-1}(x)}{x^{3/2}} = 0,$$

which is what we wanted.

4. We prove the forward direction as the proof of the reverse direction is exactly analogous. Recall that the super-Gaussianity of ψ implies that for all $\lambda \in [0, \lambda_{\max})$, we have $\psi'(\lambda) \geq \frac{2\psi(\lambda)}{\lambda}$. In particular, taking $\lambda = (\psi^*)'(u)$ for $u \in [0, u_{\max})$ for $u_{\max} := \sup_\lambda \psi'(\lambda)$ yields:

$$u = \psi'((\psi')^{-1}(u)) = \psi'((\psi^*)'(u)) \geq \frac{2\psi((\psi^*)'(u))}{(\psi^*)'(u)}.$$

Rearranging and noting that $\psi = (\psi^*)^*$ yields

$$(\psi^*)'(u) \geq \frac{2\psi((\psi^*)'(u))}{u} = \frac{2\sup_{w \in [0, u_{\max})}\{w(\psi^*)'(u) - \psi^*(w)\}}{u}$$
$$\geq \frac{2\{u(\psi^*)'(u) - (\psi^*)(u)\}}{u} = 2(\psi^*)'(u) - \frac{2\psi^*(u)}{u}.$$

Now, subtracting $2(\psi^*)'(u)$ from both sides yields

$$-(\psi^*)'(u) \geq -\frac{2\psi^*(u)}{u}.$$

Multiplying both sides by -1 furnishes the desired result.

∎

We conclude this section by discussing the slope transform, a recently proposed transform of a CGF-like function that can be used to construct time-uniform, line-crossing inequalities for martingales [31].

Definition A.4. Suppose $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is a CGF-like function. The **slope transform** associated with ψ is the mapping $\mathfrak{s} : [0, u_{\max}) \to \mathbb{R}_{\geq 0}$ given by

$$\mathfrak{s}(u) := \frac{\psi\left((\psi^*)'(u)\right)}{(\psi^*)'(u)}.$$

The slope transform, while abstract and perhaps a bit unintuitive in nature, is of great utility in optimizing our time-uniform, scalar-valued inequalities in the main body of this paper. In particular, we will leverage the following inequality in the proof of Theorem 3.1. In the following, recall that for a fixed CGF-like function $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$, we defined the quantity $u_{\max}$ as $u_{\max} := \sup_\lambda \psi'(\lambda)$. For most examples considered in this paper (in particular in the case of super-Gaussian ψ), $u_{\max} = \infty$.

Lemma A.5 (Howard et al. [31]). *Suppose $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is CGF-like, and suppose $(S_t, V_t)_{t \geq 0}$ is a sub-ψ process, per Definition 2.1. Then, for any $m > 0$, $\delta \in (0, 1)$, and any $x \in (0, mu_{\max})$, we have*

$$\mathbb{P}\left(\exists t \geq 0 : S_t \geq x + \mathfrak{s}\left(\frac{x}{m}\right)(V_t - m)\right) \leq \exp\left\{-m\psi^*\left(\frac{x}{m}\right)\right\}.$$

While the slope transform $\mathfrak{s}(u)$ may be a generally complicated function, the following upper bound allows us to greatly simplify our analysis. It is proven in Howard et al. [31].

Proposition A.6. *Suppose $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is CGF-like. Let $\mathfrak{s} : [0, u_{\max}) \to \mathbb{R}_{\geq 0}$ be the associated slope transform. Then, for any $u \in [0, u_{\max})$, $\mathfrak{s}(u) \leq u$.*

B   Proofs of Results from Sections 5 and 6

In this appendix, we provide proofs for all results related to applications of Theorem 4.1. We start by proving the regression-based results from Section 5, and then move on to proving our empirical Bernstein bound, as discussed in Section 6. We begin by providing practically-relevant examples of when the residual process $S_t = \sum_{s=1}^{t} \epsilon_s X_s$ defined in Model 5.6 is sub-ψ with variance proxy $V_t = \sum_{s=1}^{t} X_s X_s^{\top}$.

Proposition B.1. *Suppose $(X_t)_{t \geq 1}$, $(\epsilon_t)_{t \geq 1}$, and $(\mathcal{F}_t)_{t \geq 0}$ are as outlined in Model 5.1. Let us define the residual process $(S_t)_{t \geq 0}$ by $S_t := \sum_{s=1}^{t} \epsilon_s X_s$ and the covariance process $(V_t)_{t \geq 0}$ by $V_t := \sum_{s=1}^{t} X_s X_s^{\top}$. Then, $(S_t)_{t \geq 0}$ is sub-ψ with variance proxy $(V_t)_{t \geq 0}$ if either of the following conditions is satisfied.*

1. *$(\epsilon_t)_{t \geq 1}$ satisfies $\log \mathbb{E}_{t-1} \exp \{\lambda \epsilon_t\} \leq \psi_N(\lambda)$ for all $t \geq 1$ and $\lambda \geq 0$.*

2. *$\|X_t\| \leq 1$ almost surely for all $t \geq 1$ and $(\epsilon_t)_{t \geq 1}$ satisfies $\log \mathbb{E}_{t-1} \exp \{\lambda \epsilon_t\} \leq \psi(\lambda)$ for all $t \geq 1$ and $\lambda \in [0, \lambda_{\max})$, where $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is some super-Gaussian CGF-like function.*

Proof. The proof of 1 is straightforward, so we just prove 2. Observe that, from the assumption that $\psi : [0, \lambda_{\max}) \to \mathbb{R}_{\geq 0}$ is a super-Gaussian CGF-like function, we have that, for any $\lambda_1 < \lambda_2 \in [0, \lambda_{\max})$,

$$\frac{\psi(\lambda_1)}{\lambda_1^2} \leq \frac{\psi(\lambda_2)}{\lambda_2^2}.$$

Consequently, for any direction $\nu \in \mathbb{S}^{d-1}$, $\lambda \in [0, \lambda_{\max})$, and $\|x\| \leq 1$, we have

$$\frac{\psi(\lambda \langle \nu, x \rangle)}{\lambda^2 \langle \nu, x \rangle^2} \leq \frac{\psi(\lambda)}{\lambda^2}.$$

Combining this with with the CGF bound on the noise variable ϵ_t presented in Proposition B.1 (along with the assumption that $\|X_t\| \leq 1$), we have

$$\log \mathbb{E} \left(e^{\lambda \langle \nu, X_t \rangle \epsilon_t} \mid \mathcal{F}_{t-1} \right) \leq \psi(\lambda \langle \nu, X_t \rangle) \leq \langle \nu, X_t \rangle^2 \psi(\lambda),$$

where in the above we have used the fact that X_t is $\mathcal{F}_{t-1}$-measurable. This immediately yields that, for any $\lambda \in [0, \lambda_{\max})$ and $\nu \in \mathbb{S}^{d-1}$, the process $(M_t^{\lambda, \nu})_{t \geq 0}$ given by

$$M_t^{\lambda, \nu} := \exp \left\{ \lambda \sum_{s \leq t} \epsilon_s \langle \nu, X_s \rangle - \psi(\lambda) \sum_{s \leq t} \langle \nu, X_s \rangle^2 \right\} = \exp \left\{ \lambda \langle \nu, S_t \rangle - \psi(\lambda) \langle \nu, V_t \nu \rangle \right\}$$

is a non-negative supermartingale. Consequently, the scalar-valued process $(\langle \nu, S_t \rangle, \langle \nu, V_t \nu \rangle)_{t \geq 0}$ is sub-ψ for any $\nu \in \mathbb{S}^{d-1}$. Thus, by definition, the vector process $(S_t, V_t)_{t \geq 0}$ is sub-ψ in the vector-valued sense provided in Definition 2.2.

∎

We now prove Theorem 5.2.

Proof of Theorem 5.2. For a Hermitian matrix $A \in \mathbb{R}^{d \times d}$ let $A \wedge \rho I_d$ be defined equivalently to $A \vee \rho I_d$ except with the eigenvalue being set to $\gamma_i(A) \wedge \rho$ versus $\gamma_i(A) \vee \rho$. Observe that we have the identity

$$A = A \vee \rho I_d + A \wedge \rho I_d - \rho I_d. \tag{10}$$

Note that we can write the difference between our estimate and the true slope parameter as

$$
\begin{aligned}
\widehat{\theta}_t - \theta^* &= (V_t \vee \rho I_d)^{-1} \mathbf{X}_t^\top (\mathbf{X}_t \theta^* + \epsilon_{1:t}) - \theta^* \\
&= (V_t \vee \rho I_d)^{-1} (\mathbf{X}_t^\top \mathbf{X}_t \vee \rho I_d + \mathbf{X}_t^\top \mathbf{X}_t \wedge \rho I_d - \rho I_d) \theta^* + (V_t \vee \rho I_d)^{-1} S_t - \theta^* \\
&= (V_t \vee \rho I_d)^{-1} \left(\mathbf{X}_t^\top \mathbf{X}_t \wedge \rho I_d - \rho I_d \right) \theta^* + (V_t \vee \rho I_d)^{-1} S_t,
\end{aligned}
$$

where in the above we have defined the "residual process" $(S_t)_{t \geq 0}$ as $S_t := \sum_{s=1}^t \epsilon_s X_s \in \mathbb{R}^d$. In the above, the second line follows from the first by applying the equality outlined in Equation (10), the third follows from the second by recalling $V_t = \mathbf{X}_t^\top \mathbf{X}_t$ and noting a cancellation between the first and last term.

Thus, applying the triangle inequality gives us

$$
\begin{aligned}
\|(V_t \vee \rho I_d)^{1/2}(\widehat{\theta}_t - \theta^*)\| &\leq \|(V_t \vee \rho I_d)^{-1/2}(\mathbf{X}_t^\top \mathbf{X}_t \wedge \rho I_d - \rho I_d)\theta^*\| + \|(V_t \vee \rho I_d)^{-1/2} S_t\| \\
&\leq \sqrt{\rho} \|\theta^*\| \mathbb{1}_{\gamma_{\min}(\mathbf{X}_t^\top \mathbf{X}_t) < \rho} + \|(V_t \vee \rho I_d)^{-1/2} S_t\|,
\end{aligned}
$$

where the second line follows from the first via straightforward algebraic manipulation and bounding. What remains is to bound $\|(V_t \vee \rho I_d)^{-1/2} S_t\|$. But since we have assumed $(S_t)_{t \geq 0}$ is sub-ψ with variance proxy $(V_t)_{t \geq 0}$, Theorem 4.1 implies that, with probability at least $1 - \delta$, simultaneously for all $t \geq 1$, we have

$$\|(V_t \vee \rho I_d)^{-1/2} S_t\| \leq \frac{\sqrt{\gamma_{\min}(V_t \vee \rho I_d)}}{1 - \epsilon} \cdot (\psi^*)^{-1} \left(\frac{\alpha}{\gamma_{\min}(V_t \vee \rho I_d)} L_\rho(V_t) \right),$$

which finishes the proof. ∎

We now prove Corollary 5.4. The proof below is almost identical to the proof of Theorem 5.2, modulo slight modifications, so omit many details.

Proof of Corollary 5.4. Using a similar line of reasoning, we see that we have the (deterministic) inequality

$$
\begin{aligned}
\|(V_t + \rho I_d)^{1/2}(\widetilde{\theta}_t - \theta^*)\| &\leq \rho \|(V_t + \rho I_d)^{-1/2} \theta^*\| + \|(V_t + \rho I_d)^{-1/2} S_t\| \\
&\leq \sqrt{\rho} \|\theta^*\| + \|(V_t + \rho I_d)^{-1/2} S_t\|,
\end{aligned}
$$

where $(S_t)_{t \geq 0}$ is the residual process outlined in the proof of Theorem 5.2. The result now follows by noting that $(S_t, V_t)_{t \geq 0}$ is sub-ψ in the vector-sense of Definition 2.2. ∎

What remains is to prove Corollary 5.7, which concerns the estimation of model parameters in the VAR(p) model. The proof of the corollary just involves casting the estimation of model parameters in terms of the online linear regression model, i.e. Model 5.1. By the assumption that $\psi = \psi_N$, per the discussion following the statement of Theorem 5.2, it is not necessary to assume $\|X_t\| \leq 1$ for all $t \geq 1$.

Proof of Corollary 5.7. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration outlined in Model 5.6, i.e. $\mathcal{F}_t := \sigma(Y_s : -p+1 \leq s \leq t)$. Note that the $\mathbb{R}^k$-valued sequence $(X_t)_{t\geq 1}$ is $(\mathcal{F}_t)_{t\geq 1}$-predictable and the $\mathbb{R}^d$-valued noise sequence $(\epsilon_t)_{t\geq 1}$ is $(\mathcal{F}_t)_{t\geq 0}$-adapted. Further noting the identity

$$Y_t(i) = \langle \pi(i), X_t \rangle + \epsilon_t,$$

we see that we are exactly in the setting of Model 5.1. Thus, applying Theorem 5.2 yields the desired result. ∎

Lastly, we prove Theorem 6.1, which provides a self-normalized, time-uniform empirical Bernstein inequality for multivariate processes. In the proof of Theorem 6.1, we will need the following lemma, which can be extracted from the proof of Theorem 4 in Howard et al. [32], which in turn generalizes a result by Fan et al. [27].

Lemma B.2 (**Theorem 4 of Howard et al. [32]**). *Let $(X_t)_{t\geq 0}$ be a real-valued sequence of random variables adapted to some filtration $(\mathcal{F}_t)_{t\geq 0}$. Suppose that $|X_t| \leq 1/2$ almost surely for all $t \geq 1$. Then, for any $\lambda \in [0, \lambda)$, the process*

$$L_t^\lambda := \exp\left\{ \lambda \sum_{s\leq t}(X_s - \mathbb{E}_{s-1}X_s) - \psi_{E,1}(\lambda)\sum_{s\leq t}(X_s - \widehat{\mu}_{s-1})^2 \right\}$$

is a non-negative supermartingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Consequently, $\left(\sum_{s=1}^t(X_s - \mathbb{E}_{s-1}X_s)\right)_{t\geq 0}$ is sub-$\psi_{E,1}$ with variance proxy $\left(\sum_{s=1}^t(X_s - \widehat{\mu}_{s-1})^2\right)_{t\geq 0}$.

We now prove Theorem 6.1. All we need to do in the proof is check that the process $(S_t, V_t)_{t\geq 0}$ is sub-ψ_E in the sense of Definition 2.2. This boils down to checking that the projection of $(S_t, V_t)_{t\geq 0}$ onto any direction vector is sub-ψ_E in the scalar sense. With Lemma B.2 in hand, checking this condition becomes trivial.

Proof. To prove the result, it suffices to check that (S_t, V_t) is sub-$\psi_{E,1}$, per Definition 2.2. Thus, we show that, for any $\nu \in \mathbb{S}^{d-1}$, $(\langle \nu, S_t \rangle, \langle \nu, V_t\nu \rangle)_{t\geq 0}$ is sub-$\psi_{E,1}$ in the sense of Definition 2.1. Clearly, $\langle \nu, S_t \rangle \in [-1/2, 1/2]$ almost surely. Further, we have

$$\langle \nu, V_t\nu \rangle = \sum_{s=1}^t \langle \nu, (X_s - \widehat{\mu}_{s-1})(X_s - \widehat{\mu}_{s-1})^\top \nu \rangle$$

$$= \sum_{s=1}^t \langle \nu, X_s - \widehat{\mu}_{s-1} \rangle^2$$

$$= \sum_{s=1}^t (\langle \nu, X_s \rangle - \langle \nu, \widehat{\mu}_{s-1} \rangle)^2.$$

Thus, Lemma B.2 implies that $(\langle \nu, S_t \rangle, \langle \nu, V_t\nu \rangle)_{t\geq 0}$ is sub-$\psi_{E,1}$, so the first claim follows. The second claim follows from Theorem 4.1. Finally, for any $\lambda \in [0, 1)$,

$$\psi_{E,1}(\lambda) = -\log(1-\lambda) - \lambda \leq \frac{\lambda^2}{2(1-\lambda)} =: \psi_{G,1}(c),$$

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so $(S_t, V_t)_{t \geq 0}$ is sub-$\psi_{G,1}$ also. Noting that

$$(\psi_{G,1}^*)^{-1}(u) = \sqrt{2u} + u$$

yields the final claim. Proofs of these two facts surrounding $\psi_{E,1}$ and $\psi_{G,1}$ can be found in Boucheron et al. [11]. ∎

C   Proofs of Technical Lemmas

In this section, we provides proofs for the technical lemmas used in proving the main results of this paper. We start by proving Lemma 7.1, which is used in the proof of Theorem 4.1.

Proof of Lemma 7.1. Let $\nu \in \mathbb{S}^{d-1}$ be arbitrary and let $\omega \in \mathbb{S}^{d-1}$ be the unique vector satisfying (7). By the definition of ω and π_T, we have

$$
\begin{aligned}
\|\nu - \pi_T(\nu)\| &= \left\| \frac{T^{1/2}\omega}{\|T^{1/2}\omega\|} - \frac{T^{1/2}\pi(\omega)}{\|T^{1/2}\pi(\omega)\|} \right\| \\
&\leq \max \left\{ \left\| \frac{T^{1/2}\omega}{\|T^{1/2}\omega\|} - \frac{T^{1/2}\pi(\omega)}{\|T^{1/2}\omega\|} \right\|, \left\| \frac{T^{1/2}\omega}{\|T^{1/2}\pi(\omega)\|} - \frac{T^{1/2}\pi(\omega)}{\|T^{1/2}\pi(\omega)\|} \right\| \right\} \\
&\leq \frac{\gamma_{\max}(T^{1/2})}{\|T^{1/2}\omega\| \wedge \|T^{1/2}\pi(\omega)\|} \|\omega - \pi(\omega)\| \leq \sqrt{\kappa}\epsilon.
\end{aligned}
$$

Above, the second inequality follows from pulling out the denominator in each term of the maximum and bounding $\|T^{1/2}(\omega - \pi(\omega))\| \leq \gamma_{\max}(T^{1/2})\|\omega - \pi(\omega)\|$ and the last inequality follows as $\|T^{1/2}\omega\| \wedge \|T^{1/2}\pi(\omega)\| \geq \gamma_{\min}(T^{1/2})$, and $\|\omega - \pi(\omega)\| \leq \epsilon$ by definition of projection onto a cover. The first inequality follows from a simple calculation. To elaborate, assume $\|T^{1/2}\omega\| \neq \|T^{1/2}\pi(\omega)\|$, as in the case of equality there is nothing to prove in the inequality. Notice that if $\|T^{1/2}\omega\| < \|T^{1/2}\pi(\omega)\|$, then $q := T^{1/2}\omega/\|T^{1/2}\omega\|$ lies on the surface of the unit ball, and $p := T^{1/2}\pi(\omega)/\|T^{1/2}\omega\|$ lies outside of the unit ball (i.e. has norm greater than 1). The projection of p onto the unit ball is exactly $T^{1/2}\pi(\omega)/\|T^{1/2}\pi(\omega)\|$, which is closer to q than p since projections onto convex sets decrease Euclidean distance to all points. The maximum above comes from handling the case $\|T^{1/2}\omega\| > \|T^{1/2}\pi(\omega)\|$, which is analogous. This shows the desired result. ∎

We now prove Lemma 4.2, which is leveraged in the proof of Corollary 4.6 and Corollary 4.3.

Proof of Lemma 4.2. We start by providing an upper bound on the proper ϵ-covering number for $\mathbb{S}_\infty^{d-1} := \{x \in \mathbb{R}^d : \|x\|_\infty := \max_{j \in [d]} |x_j| = 1\}$. Note that we can write

$$\mathbb{S}_\infty^{d-1} = \bigcup_{i=1}^d F_i^+ \cup F_i^-,$$

where $F_i^+ := \{x \in \mathbb{R}^d : x_i = 1, \|x_{-i}\|_\infty \leq 1\}$ and $F_i^- := \{x \in \mathbb{R}^d : x_i = -1, \|x_{-i}\|_\infty \leq 1\}$, where $x_{-i} \in R^{d-1}$ is the vector x with the ith component omitted. For any $i \in [d]$, $s \in$

$\{+, -\}$, there is a natural isometry between F_i^s and the $(d-1)$-dimensional ℓ_∞ ball defined as $\mathbb{B}_{d-1}^\infty := \{x \in \mathbb{R}^{d-1} : \|x\|_\infty \leq 1\}$ given by $x \in \mathbb{R}^d \mapsto x_{-i} \in \mathbb{R}^{d-1}$. In particular, this implies the proper ϵ-covering number of F_i^s under the ℓ_2-norm is bounded as

$$N\left(F_i^s, \epsilon, \| \cdot \|\right) = N\left(\mathbb{B}_{d-1}^\infty, \epsilon, \| \cdot \|\right) \leq \frac{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1}^\infty)}{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1})}\left(\frac{3}{\epsilon}\right)^{d-1},$$

where the last inequality follows from Lemma 5.7 of Wainwright [57]. From this, we see that we have the bound

$$N\left(\mathbb{S}_\infty^{d-1}, \epsilon, \| \cdot \|\right) \leq 2d\frac{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1}^\infty)}{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1})}\left(\frac{3}{\epsilon}\right)^{d-1} = C_d\left(\frac{3}{\epsilon}\right)^{d-1},$$

where we have summed over the $2d$ different $(d-1)$-dimensional faces $F_1^+, F_1^-, \ldots, F_d^+, F_d^-$ and defined the constant $C_d := 2d\frac{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1}^\infty)}{\mathrm{Vol}_{d-1}(\mathbb{B}_{d-1})}$.

Let K now denote a minimal proper ϵ-covering of $\mathbb{S}_\infty^{d-1}$, and let $\pi : \mathbb{S}_\infty^{d-1} \to K$ denote the projection onto the covering. Further, let $p : \mathbb{R}^d \to \mathbb{B}_d$ denote the ℓ_2 projection onto the unit ball. We claim that the set $K' := \{p(x) : x \in K\}$ is a proper ϵ-covering of $\mathbb{S}^{d-1}$ under the ℓ_2 norm. The fact that $p(x) \in \mathbb{S}^{d-1}$ is immediate. Next, note that for any $y \in \mathbb{S}^{d-1}$, there is some $x \in \mathbb{S}_\infty^{d-1}$ such that $p(x) = y$. Observe that $z := p(\pi(x)) \in K'$. Since p is an ℓ_2 projection, we know that

$$\|y - z\| = \|p(x) - p(\pi(x))\| \leq \|x - \pi(x)\| \leq \epsilon,$$

so we have shown that K' is a proper ϵ-covering. ∎

D Figures

(a) $\alpha = 1.01$

(b) $\alpha = 1.05$

(c) $\alpha = 1.25$

(d) $\alpha = 1.5$
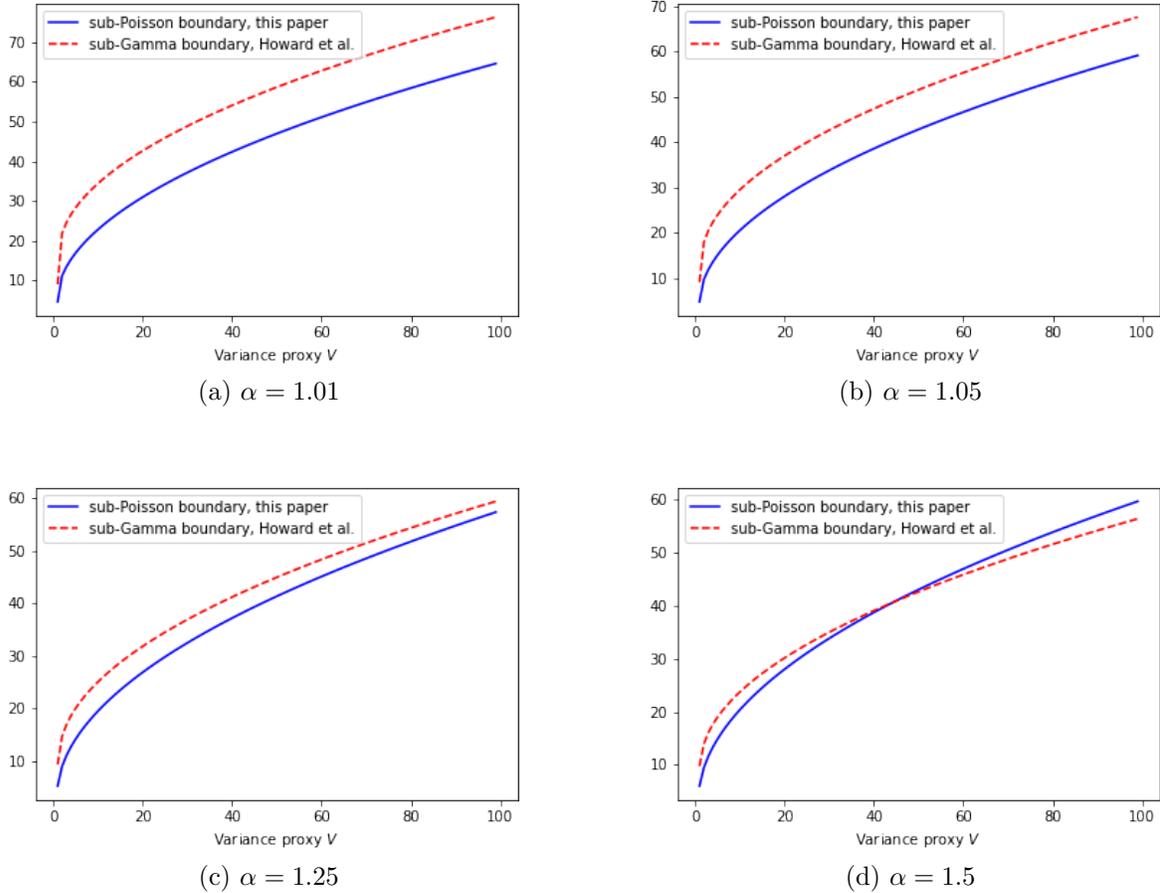
Figure 1: Comparing the boundary of Theorem 3.1 in the case $\psi = \psi_{P,c}$ with the boundary of Theorem 1 in Howard et al. [32], recapped in (1). Note that to apply the boundary of Howard et al. [32], we need to leverage the fact that a sub-$\psi_{P,c}$ process $(S_t)_{t \geq 0}$ is also sub-$\psi_{G,c}$ with the same variance proxy $(V_t)_{t \geq 0}$. We have made the parameter selection $c = 1$, $\delta = 0.01$, $\rho = 1$, and $h(k) = (1+k)^2 \zeta(2)$, and have correspondingly varied α over several values. We see that for reasonably small choices of intrinsic time spacing $\alpha > 1$, our boundary is tighter than that of Howard et al. [32]. Thus, we see that although a sub-$\psi_{P,c}$ process can be viewed as a sub-$\psi_{G,c}$ process, this conversion introduces looseness, making our time-uniform concentration result generally preferable in this setting.
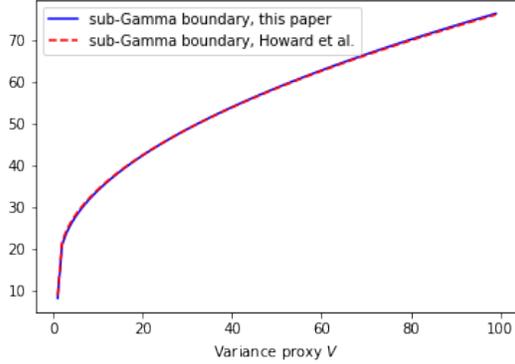
(a) $\alpha = 1.01$
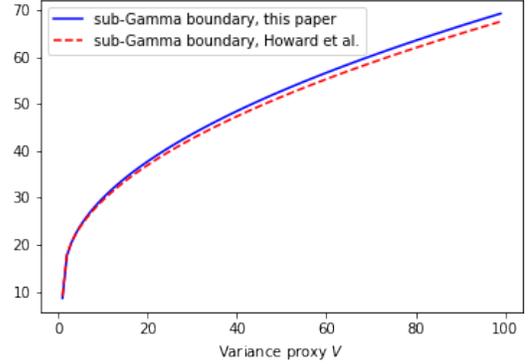
(b) $\alpha = 1.05$
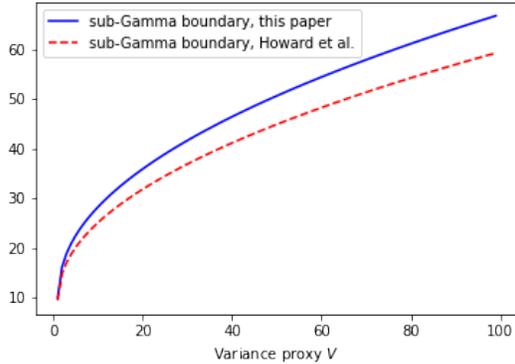
(c) $\alpha = 1.25$
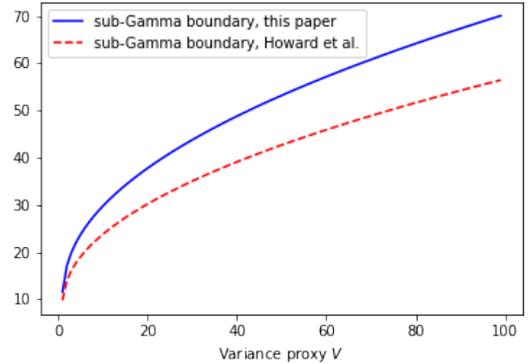
(d) $\alpha = 1.5$

Figure 2: Comparing the boundary of Theorem 3.1 in the case $\psi = \psi_{G,c}$ with the boundary of Howard et al. [32] (presented in Equation 1). We have made the parameter selection $c = 1$, $\delta = 0.01$, $\rho = 1$, and $h(k) = (1 + k)^2 \zeta(2)$, and have correspondingly varied α over several values. As expected from our discussion, our boundary is looser than that of Howard et al. [32] for all values of α, with the gap between the boundaries vanishing as the geometric spacing α of variance/intrinsic time is decreased towards 1. Since $\alpha = 1.01$ or $\alpha = 1.05$ are reasonable choices for applying our concentration inequalities, our bounds are just as applicable as those of Howard et al. [32] even in the sub-Gamma setting.
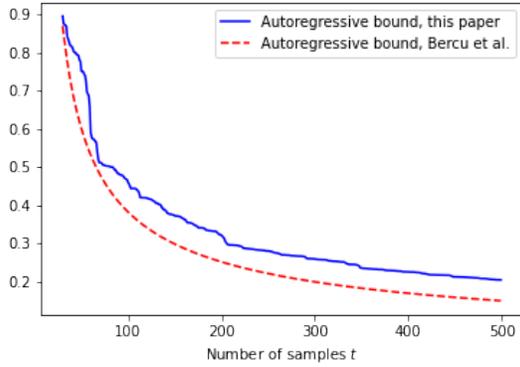
(a) Fact 5.8 vs. Corollary 5.7 *without* union bound
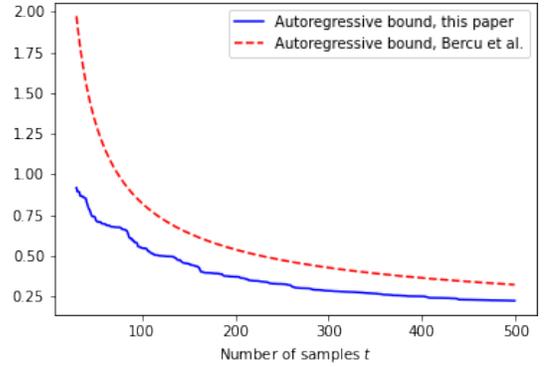
(b) Fact 5.8 vs. Corollary 5.7 *with* union bound

Figure 3: A comparison of the bounds on $|\widehat{a}_t - a|$ provided by Fact 5.8 and Corollary 5.7. In plotting both bounds, we have fixed the failure probability as $\delta = 0.01$. We have numerically solved for x such that the right hand side of Fact 5.8 is equal to the target failure probability. When applying Corollary 5.7, we have set $\alpha = 1.5, h(k) = (1+k)^2\zeta(2), \rho = 1$, and note that dependence on ϵ and β can be removed in the univariate case. In Subfigure 3(a), we plot Fact 5.8 point-wise (i.e. we set the failure probability to be δ for each sample size t), and in Subfigure 3(b), we take a union bound over samples, setting the failure probability to be $\frac{6\delta}{t^2\pi^2}$ for each t.